An Isometrical $\mathbb{CP}^n$-Theorem [1]

Xiaole Su, Hongwei Sun, Yusheng Wang[2]

Abstract. Let M^n ($n \geq 3$) be a complete Riemannian manifold with $\sec_M \geq 1$, and let $M_i^{n_i}$ ($i = 1, 2$) be two comlplete totally geodesic submanifolds in M. We prove that if $n_1 + n_2 = n - 2$ and if the distance $|M_1 M_2| \geq \frac{\pi}{2}$, then M_i is isometric to $\mathbb{S}^{n_i}/\mathbb{Z}_h$, $\mathbb{CP}^{\frac{n_i}{2}}$ or $\mathbb{CP}^{\frac{n_i}{2}}/\mathbb{Z}_2$ with the canonical metric when $n_i > 0$, and thus M is isometric to $\mathbb{S}^n/\mathbb{Z}_h$, $\mathbb{CP}^{\frac{n}{2}}$ or $\mathbb{CP}^{\frac{n}{2}}/\mathbb{Z}_2$ except possibly when $n = 3$ and M_1 (or M_2) $\overset{\text{iso}}{\cong} \mathbb{S}^1/\mathbb{Z}_h$ with $h \geq 2$ or $n = 4$ and M_1 (or M_2) $\overset{\text{iso}}{\cong} \mathbb{RP}^2$.

Key words. Rigidity, positive sectional curvature, totally geodesic submanifolds

Mathematics Subject Classification (2000): 53-C20.

0 Introduction

Let M be a complete, simply connected Riemannian manifold with $\sec_M \geq 1$.

$$\textit{Under what conditions is } M \textit{ isometric to } \mathbb{S}^n \textit{ or a projective space } \mathbb{KP}^n? \tag{0.1}$$

Here, $\mathbb{S}^n$ is the unit sphere and $\mathbb{KP}^n$ is endowed with the canonical metric, where $\mathbb{K} = \mathbb{C}, \mathbb{H}$ or $\mathbb{C}a$ and $n \leq 2$ if $\mathbb{K} = \mathbb{C}a$, which satisfies

$$\sec_{\mathbb{S}^n} \equiv 1 \text{ and } \operatorname{diam}(\mathbb{S}^n) = \pi, \text{ and } 1 \leq \sec_{\mathbb{KP}^n} \leq 4 \text{ and } \operatorname{diam}(\mathbb{KP}^n) = \frac{\pi}{2}.$$

This question draws lots of attention from geometrists. Note that "$\sec_M \geq 1$" implies that the diameter $\operatorname{diam}(M) \leq \pi$. Toponogov proved that *if* $\operatorname{diam}(M) = \pi$ *(here that M is simply connected is not needed), then M is isometric to* $\mathbb{S}^n$ (Maximal Diameter Theorem). And Berger proved that *if* $1 \leq \sec_M \leq 4$, *then either* $\operatorname{diam}(M) > \frac{\pi}{2}$ *and M is homeomorphic to a sphere, or* $\operatorname{diam}(M) = \frac{\pi}{2}$ and M is isometric to $\mathbb{S}^n(\frac{1}{2})$ or a $\mathbb{KP}^n$ (Minimal Diameter Theorem, [CE]). Afterwards, Grove-Shiohama proved that *if* $\operatorname{diam}(M) > \frac{\pi}{2}$ *(here that M is simply connected is not needed), then M is homeomorphic to a sphere* ([GS]). Inspired by these, Gromoll-Grove, Wilhelm and Wilking proved step by step that *if* $\operatorname{diam}(M) = \frac{\pi}{2}$, *then M is either homeomorphic to a sphere, or isometric to a* $\mathbb{KP}^n$ ($\frac{\pi}{2}$-Diameter Rigidity Theorem, [GG1,2], [W], [Wi1]).

Note that the isometric classification in the $\frac{\pi}{2}$-Diameter Rigidity Theorem is on the premise that M is not homeomorphic to a sphere. The present paper aims to give "purely isometric" answers to question (0.1) (as Toponogov's and Berger's above).

A basic fact is that $\mathbb{S}^n$ has a join structure, i.e.,

$$\mathbb{S}^n = \mathbb{S}^{n_1} * \mathbb{S}^{n_2}, \tag{0.2}$$

[1]Supported by NSFC 11471039.

[2]The corresponding author (E-mail: wyusheng@bnu.edu.cn).

where $n_1+n_2=n-1$, and each $\mathbb{S}^{n_i}$ is totally geodesic in $\mathbb{S}^n$, and the distance

$$|p_1p_2|=\frac{\pi}{2} \text{ for all } p_i\in\mathbb{S}^{n_i}.$$

Similarly, we can define a spherical join of two Alexandrov spaces X_i with curvature ≥ 1 (including Riemannian manifolds with $\sec\geq 1$), $X_1 * X_2$, which is also an Alexandrov space with curvature ≥ 1 ([BGP]). And, in $X_1 * X_2$, X_i is convex[3] with $\dim(X_1)+\dim(X_2)=\dim(X)-1$, and $|p_1p_2|=\frac{\pi}{2}$ for all $p_i\in X_i$.

Inspired by this, Rong-Wang obtains the following rigidity theorem.

Theorem 0.1 ([RW]). *Let X be a compact Alexandrov space with curvature ≥ 1 and of dimension n, and let X_i be its two compact convex subsets with empty boundary and of dimension n_i. If $|X_1X_2|\triangleq\min\{|p_1p_2|\mid p_i\in X_i\}\geq\frac{\pi}{2}$, then $n_1+n_2\leq n-1$, and equality implies that X is isometric to a spherical join modulo a finite group.*

In Riemannian case we have the following corollary.

Corollary 0.2 ([RW]). *Let M^n be a complete Riemannian manifold with $\sec_M\geq 1$, and let $M_i^{n_i}$ be its two complete totally geodesic submanifolds. If $|M_1M_2|\geq\frac{\pi}{2}$, then $n_1+n_2\leq n-1$, and equality implies that there is a finite group Γ such that M_i is isometric to $\mathbb{S}^{n_i}/\Gamma$ and M is isometric to $\mathbb{S}^n/\Gamma$.*

Naturally, the next step is to consider the case where $n_1+n_2=n-2$.

Conjecture 0.3. *For X and X_i in Theorem 0.1, if $|X_1X_2|\geq\frac{\pi}{2}$ and $n_1+n_2=n-2$, then either X_i belong to a compact convex subset of dimension $n-1$ in X, or X is isometric to a spherical join modulo a 1-dimensional Lie group (with finite components).*

Remark 0.4. Note that this conjecture is trivial when n_1 or $n_2=0$. The reason is that if $n_i=0$ and X_i has an empty boundary, then it is our convention that X_i consists of two points with distance π. In this case, X is isometric to a spherical join, and isometric to a unit sphere in Riemannian case (cf. [RW]).

In general case, Conjecture 0.3 will be much harder than Theorem 0.1. In the present paper, the main result asserts that the conjecture is true in Riemannian case.

First, let's focus on the complex projective space $\mathbb{CP}^m$ as an example satisfying the conjecture. In (0.2), we assume that $n=2m+1$ and $n_i=2m_i+1$. We know that S^1 can act on $\mathbb{S}^n$ freely and isometrically and preserve each $\mathbb{S}^{n_i}$ such that

$$\mathbb{S}^n/S^1\overset{\text{iso}}{\cong}\mathbb{CP}^m \text{ and } \mathbb{S}^{n_i}/S^1\overset{\text{iso}}{\cong}\mathbb{CP}^{m_i}. \tag{0.3}$$

Note that $\mathbb{CP}^{m_i}$ is totally geodesic in $\mathbb{CP}^m$ with

$$|q_1q_2|=\frac{\pi}{2} \text{ for all } q_i\in\mathbb{CP}^{m_i},$$

and that $\dim(\mathbb{CP}^{m_1})+\dim(\mathbb{CP}^{m_2})=\dim(\mathbb{CP}^m)-2$.

[3]We say that a subset A is convex in X if, for any $x,y\in A$, there is a minimal geodesic (in X) jointing x with y which belongs to A.

Now, let's formulate our main result in this paper.

Main Theorem. *Let M^n ($n \geq 3$) be a complete Riemannian manifold with* $\sec_M \geq 1$*, and let $M_i^{n_i}$ be its two complete totally geodesic submanifolds. If $|M_1M_2| \geq \frac{\pi}{2}$ and $n_1 + n_2 = n - 2$, then M_i is isometric to $\mathbb{S}^{n_i}/\mathbb{Z}_h$, $\mathbb{CP}^{\frac{n_i}{2}}$ or $\mathbb{CP}^{\frac{n_i}{2}}/\mathbb{Z}_2$ when $n_i > 0$, and thus M is isometric to $\mathbb{S}^n/\mathbb{Z}_h$, $\mathbb{CP}^{\frac{n}{2}}$ or $\mathbb{CP}^{\frac{n}{2}}/\mathbb{Z}_2$ except possibly when $n = 3$ and M_1 (or M_2) is isometric to $\mathbb{S}^1/\mathbb{Z}_h$ with $h \geq 2$ or $n = 4$ and M_1 (or M_2) is isometric to $\mathbb{RP}^2$.*

Here, we make a convention that M_i contains only one point if $n_i = 0$, and $\mathbb{S}^1$ is the circle with perimeter 2π. And one can refer to A.1 in Appendix for the construction of $\mathbb{CP}^{\frac{n}{2}}/\mathbb{Z}_2$ ($= \mathbb{S}^{n+1}/G$, where G is a 1-dimensional Lie group with two components).

Remark 0.5. Note that together with Remark 0.4, the Main Theorem implies Conjecture 0.3 in Riemannian case.

Remark 0.6. On the Main Theorem, we have the following notes.
(0.6.1) If $\sec_{M_i} \not\equiv 1$ ($n_i > 1$) for $i = 1$ or 2 or if there is an infinite number of minimal geodesics between some $p_1 \in M_1$ and $p_2 \in M_2$, then $M \overset{\text{iso}}{\cong} \mathbb{CP}^{\frac{n}{2}}$ or $\mathbb{CP}^{\frac{n}{2}}/\mathbb{Z}_2$ (see Proposition 3.1 and Lemma 3.14), and that $M \overset{\text{iso}}{\cong} \mathbb{CP}^{\frac{n}{2}}/\mathbb{Z}_2$ occurs only when $\frac{n_i}{2}$ and $\frac{n}{2}$ are all odd.
(0.6.2) If n is odd, then $M \overset{\text{iso}}{\cong} \mathbb{S}^n/\mathbb{Z}_h$, and $h \geq 3$ implies that n_1 or $n_2 = 0$; i.e., if in addition $n_1, n_2 > 0$, then $M \overset{\text{iso}}{\cong} \mathbb{S}^n$ or $\mathbb{RP}^n$ (see Lemma 3.3 and 3.8).
(0.6.3) Suppose that M is simply connected. If $n \geq 5$, then $M \overset{\text{iso}}{\cong} \mathbb{S}^n$ or $\mathbb{CP}^{\frac{n}{2}}$; if $n = 4$ (resp. $n = 3$), either $M \overset{\text{iso}}{\cong} \mathbb{S}^4$ or $\mathbb{CP}^2$ (resp. $\mathbb{S}^3$), or one of $M_i \overset{\text{iso}}{\cong} \mathbb{RP}^2$ (resp. $\mathbb{S}^1/\mathbb{Z}_h$ with $h \geq 2$) (in this case, M is homeomorphic to $\mathbb{S}^4$ (resp. $\mathbb{S}^3$) by the $\frac{\pi}{2}$-Diameter Rigidity Theorem).

Based on (0.6.3), we have the following two questions.

Problem 0.7. *Can $\mathbb{RP}^2$ be embedded isometrically into M^4 as a totally geodesic submanifold, where M^4 is a complete Riemannian manifold with* $\sec \geq 1$ *and is homeomorphic to $\mathbb{S}^4$?*

Problem 0.8. *In the Main Theorem, if $n \geq 8$ and $n_1 + n_2 \geq n - 4$ (resp. $n \geq 16$ and $n_1 + n_2 \geq n - 8$), is M isometric to $\mathbb{S}^n$, $\mathbb{CP}^{\frac{n}{2}}$ or $\mathbb{HP}^{\frac{n}{4}}$ (resp. $\mathbb{S}^n$, $\mathbb{CP}^{\frac{n}{2}}$, $\mathbb{HP}^{\frac{n}{4}}$ or $\mathbb{C}a\mathbb{P}^2$) when M is simply connected?*

We can use the approach to the Main Theorem to discuss Problem 0.8, but some essential difficulties will arise.

It seems that there is some overlap between the Main Theorem and the $\frac{\pi}{2}$-Diameter Rigidity Theorem. We will end this section by pointing out the main difference between them by comparing the key points in their proofs.

Remark 0.9. (0.9.1) *The key point to the $\frac{\pi}{2}$-Diameter Rigidity Theorem*:
To the $\frac{\pi}{2}$-Diameter Rigidity Theorem, an important fact is that $B' = B'''$ for any

compact subset $B \subset M$, where $B' = \{p \in M|\ |pb| = \frac{\pi}{2}\ \forall\ b \in B\}$ which is convex in M. This is guaranteed by $\sec_M \geq 1$ and $\mathrm{diam}(M) = \frac{\pi}{2}$ via Toponogov's Comparison Theorem (see Theorem 1.1 below). In [GG1], B' and B'' are called a pair of dual sets. Then either both B' and B'' are contractible, and in this case M is homeomorphic to a sphere; or both B' and B'' are totally geodesic submanifolds, and any $p \in M$ belongs to some minimal geodesic $[q_1q_2]$ with $q_1 \in B'$ and $q'' \in B''$, and then M is isometric to a $\mathbb{KP}^n$ (the proof involves several big classification theorems ([GG2], [Wi1]), e.g. Bott-Samelson's Theorem in [B]). In any case, the key fact that B' and B'' are dual to each other plays a crucial role.

(0.9.2) *The key point to the Main Theorem*:

In our Main Theorem, we in fact have that $|p_1p_2| = \frac{\pi}{2}$ for all $p_i \in M_i$ (see Corollary 2.2 below). Hence, if M_1 and M_2 are dual to each other, then we can use the approach to the $\frac{\pi}{2}$-Diameter Rigidity Theorem to prove that M is isometric to $\mathbb{CP}^{\frac{n}{2}}$ or $\mathbb{CP}^{\frac{n}{2}}/\mathbb{Z}_2$. Indeed, if $n_1 > 0$ and $n_2 > 0$, then we can prove that either $M \overset{\text{iso}}{\cong} \mathbb{S}^n$ or $\mathbb{RP}^n$, or M_1 and M_2 are dual to each other (see Proposition A.4 in Appendix). However, if one of $n_i = 0$, we cannot see that M_1 and M_2 are dual to each other. An important reason is that, for any one of M_i with $n_i > 0$,

$$\left\{p \in M|\ |pM_i| \geq \frac{\pi}{2}\right\} = \left\{p \in M|\ |pp_i| = \frac{\pi}{2}\ \forall\ p_i \in M_i\right\}$$

(see Lemma 2.1 below); but this may not be true if $n_i = 0$ (i.e. M_i is a single point). Therefore, the really challenging case to the Main Theorem is where M_1 or M_2 is a single point. Our proof for it, which also fits the case where $n_1 > 0$ and $n_2 > 0$, is based on an easy observation that $\lambda_{p_1p_2} = \lambda_{p_1'p_2'}$ for all $p_i, p_i' \in M_i$, where $\lambda_{p_1p_2}$ denotes the number of all minimal geodesics between p_1 and p_2 (see Corollary 2.5 below). If $\lambda_{p_1p_2}$ is finite (resp. infinite) and $n_i > 0$, then we can prove that M_i is isometric to $\mathbb{S}^{n_i}/\mathbb{Z}_h$ (resp. $\mathbb{CP}^{\frac{n_i}{2}}$ or $\mathbb{CP}^{\frac{n_i}{2}}/\mathbb{Z}_2$). In proving that M_i is isometric to $\mathbb{CP}^{\frac{n_i}{2}}$ or $\mathbb{CP}^{\frac{n_i}{2}}/\mathbb{Z}_2$, we do not use any big classification theorem involved in the proof of the $\frac{\pi}{2}$-Diameter Rigidity Theorem, i.e. our method is quite different from that in [GG1] and [Wi1].

1 Toponogov's Comparison Theorem

In this paper, we always let $[pq]$ denote a minimal geodesic between p and q in a Riemannian manifold, and let $|pq|$ denote the distance between p and q. Now, we give the main tool of the paper—Toponogov's Comparison Theorem.

Theorem 1.1 ([P], [GM]). *Let M be a complete Riemannian manifold with $\sec_M \geq \kappa$, and let $\mathbb{S}^2_\kappa$ be the complete, simply connected 2-manifold of curvature κ.*

(i) *To any $p \in M$ and $[qr] \subset M$, we associate $\tilde{p}$ and $[\tilde{q}\tilde{r}]$ in $\mathbb{S}^2_\kappa$ with $|\tilde{p}\tilde{q}| = |pq|, |\tilde{p}\tilde{r}| = |pr|$ and $|\tilde{r}\tilde{q}| = |rq|$. Then for any $s \in [qr]$ and $\tilde{s} \in [\tilde{q}\tilde{r}]$ with $|qs| = |\tilde{q}\tilde{s}|$, we have that $|ps| \geq |\tilde{p}\tilde{s}|$.*

(ii) *To any* $[qp]$ *and* $[qr]$ *in* M*, we associate* $[\tilde{q}\tilde{p}]$ *and* $[\tilde{q}\tilde{r}]$ *in* $\mathbb{S}^2_\kappa$ *with* $|\tilde{q}\tilde{p}| = |qp|$, $|\tilde{q}\tilde{r}| = |qr|$ *and* $\angle\tilde{p}\tilde{q}\tilde{r} = \angle pqr$*. Then we have that* $|\tilde{p}\tilde{r}| \geq |pr|$*.*

(iii) *If the equality in* (ii) *(resp. in* (i) *for some* s *in the interior part of* $[qr]$*) holds, then there exists a* $[pr]$ *(resp.* $[pq]$ *and* $[pr]$*) such that the triangle formed by* $[qp]$, $[qr]$ *and* $[pr]$ *bounds a surface which is convex and can be embedded isometrically into* $\mathbb{S}^2_\kappa$*.*

2 Preliminaries

In this section, all M_i $(i = 1, 2)$ and M are the manifolds in the Main Theorem.

By (ii) of Theorem 1.1, one can prove the following interesting lemma.

Lemma 2.1 ([Ya])**.** *Let* N *be a complete Riemannian manifold with* $\sec_M \geq 1$*, and let* L *be a complete totally geodesic submanifold in* N *with* $\dim(L) \geq 1$*. Then* $L^{\geq\frac{\pi}{2}} = L^{=\frac{\pi}{2}}$*.*

In this lemma, $L^{\geq\frac{\pi}{2}}$ (resp. $L^{=\frac{\pi}{2}}$) denotes the set $\{p \in N | |px| \geq \frac{\pi}{2}\ \forall\ x \in L\}$ (resp. $\{p \in N | |px| = \frac{\pi}{2}\ \forall\ x \in L\}$). (This lemma has an Alexandrov version in [Ya], and one can refer to [SW] for its detailed proof.) From Lemma 2.1, we can draw an immediate corollary (which is fundamental and important to the paper).

Corollary 2.2. *Under the conditions of the Main Theorem, we have that*

$$|p_1p_2| = \frac{\pi}{2} \text{ for any } p_1 \in M_1 \text{ and } p_2 \in M_2. \tag{2.1}$$

In this paper, we will let $\uparrow_p^q$ denote the unit tangent vector at p of a given (minimal geodesic) $[pq]$ (which is also called the direction from p to q along $[pq]$); and let Σ_pM denote the set of all unit tangent vectors in T_pM. By (2.1) and (ii) of Theorem 1.1, for any $[p_1p_2]$ and $[p_1p_2']$ with $p_1 \in M_1$ and $p_2, p_2' \in M_2$, we have that

$$|\uparrow_{p_1}^{p_2}\uparrow_{p_1}^{p_2'}| \geq |p_2p_2'|. \tag{2.2}$$

Now, we fix an arbitrary $[p_1p_2]$ with $p_i \in M_i$. By Corollary 2.2, we conclude that

$$\uparrow_{p_2}^{p_1} \in (\Sigma_{p_2}M_2)^{=\frac{\pi}{2}} \subset \Sigma_{p_2}M. \tag{2.3}$$

Similarly, we have that

$$\uparrow_{p_1}^{p_2} \in (\Sigma_{p_1}M_1)^{=\frac{\pi}{2}} \subset \Sigma_{p_1}M, \tag{2.4}$$

where $(\Sigma_{p_1}M_1)^{=\frac{\pi}{2}} = \Sigma_{p_1}M$ if $n_1 = 0$. By (iii) of Theorem 1.1, (2.3) and (2.1) imply the following easy fact.

Lemma 2.3. *For any given* $[p_1p_2]$ *with* $p_i \in M_i$ *and* $[p_2p_2'] \subset M_2$*, there exists a* $[p_1p_2']$ *such that the triangle formed by* $[p_1p_2]$*,* $[p_2p_2']$ *and* $[p_1p_2']$ *bounds a surface which is convex in* M *and can be embedded isometrically into the unit sphere* $\mathbb{S}^2$*.*

For convenience, we call such a surface in Lemma 2.3 a *convex spherical surface*. Now, based on Lemma 2.3, we give another important observation.

Lemma 2.4. *For any given $p_1 \in M_1$ and $[p_2p_2'] \subset M_2$, there is a 1-1 map*

$$\iota : \{\textit{minimal geodesics between } p_1 \textit{ and } p_2\} \to \{\textit{minimal geodesics between } p_1 \textit{ and } p_2'\}$$

such that, for any $[p_1p_2]$, $\iota([p_1p_2])$ is the unique minimal geodesic such that the triangle formed by $[p_1p_2]$, $\iota([p_1p_2])$ and $[p_2p_2']$ bounds a convex spherical surface.

Note that, in this lemma, if there is a sequence of minimal geodesics $[p_1p_2]_j$ (between p_1 and p_2) with $\lim\limits_{j\to\infty}[p_1p_2]_j \to [p_1p_2]$, then it is not hard to see that

$$\lim_{j\to\infty} \iota([p_1p_2]_j) \to \iota([p_1p_2]). \tag{2.5}$$

And this lemma has an almost immediate corollary.

Corollary 2.5. *Under the conditions of the Main Theorem, we have that*

$$\lambda_{p_1p_2} = \lambda_{p_1'p_2'}\ \forall\ p_i, p_i' \in M_i,$$

where $\lambda_{p_1p_2}$ denotes the number of $\{$minimal geodesics between p_1 and $p_2\}$.

Proof of Lemma 2.4. For any given $[p_1p_2]$, by Lemma 2.3, there is a $[p_1p_2']$ such that the triangle formed by $[p_1p_2]$, $[p_1p_2']$ and $[p_2p_2']$ bounds a convex spherical surface D. Note that D determines a minimal geodesic $[\uparrow_{p_2}^{p_1}\uparrow_{p_2}^{p_2'}]$ of length $\frac{\pi}{2}$ in $\Sigma_{p_2}M$ (which is isometric to $\mathbb{S}^{n-1}$, so there is a unique minimal geodesic between $\uparrow_{p_2}^{p_1}$ and $\uparrow_{p_2}^{p_2'}$). Hence, for the given $[p_1p_2]$, such a $[p_1p_2']$ is unique and vice versa, so the lemma follows. $\square$

In the following, for any fixed $p_1 \in M_1$, we will discuss the multi-valued map

$$f_{p_1} : M_2 \to (\Sigma_{p_1}M_1)^{=\frac{\pi}{2}} \text{ defined by } p_2 \mapsto \Uparrow_{p_1}^{p_2},$$

where $\Uparrow_{p_1}^{p_2}$ denotes the set of unit tangent vectors at p_1 of all minimal geodesics between p_1 and p_2. Note that f_{p_1} is well defined because of (2.4). Obviously, $(\Sigma_{p_1}M_1)^{=\frac{\pi}{2}} = \mathbb{S}^{n_2+1}$. For convenience, we let $\mathbb{S}_{p_1}^{n_2+1}$ denote $(\Sigma_{p_1}M_1)^{=\frac{\pi}{2}}$.

We first note that, for any $[p_1p_2]$ with $p_i \in M_i$, by Lemma 2.4 we can define a map

$$f_{[p_1p_2]} : M_2 \to \mathbb{S}_{p_1}^{n_2+1}$$

by $p_2 \mapsto \uparrow_{p_1}^{p_2}$ and $p_2' \mapsto$ some $\uparrow_{p_1}^{p_2'}$ for any other $p_2' \in M_2$ such that

$$|f_{[p_1p_2]}(p_2)f_{[p_1p_2]}(p_2')| = |p_2p_2'|, \tag{2.6}$$

and that

$$f_{[p_1p_2]}([p_2p_2']) \text{ is a minimal geodesic } [\uparrow_{p_1}^{p_2}\uparrow_{p_1}^{p_2'}] \text{ in } \mathbb{S}_{p_1}^{n_2+1} \tag{2.7}$$

if $[p_2p_2']$ is the unique minimal geodesic between p_2 and p_2'. Then we can define a "differential" map (cf. [RW])

$$\mathrm{d}f_{[p_1p_2]} : \Sigma_{p_2}M_2 \to \Sigma_{\uparrow_{p_1}^{p_2}}\mathbb{S}_{p_1}^{n_2+1} \text{ by } \uparrow_{p_2}^{p_2'} \mapsto \uparrow_{\uparrow_{p_1}^{p_2}}^{\uparrow_{p_1}^{p_2'}}.$$

Note that $\Sigma_{p_2}M_2 \overset{\text{iso}}{\cong} \mathbb{S}^{n_2-1}$ and $\Sigma_{\uparrow_{p_1}^{p_2}}\mathbb{S}_{p_1}^{n_2+1} \overset{\text{iso}}{\cong} \mathbb{S}^{n_2}$. About $\mathrm{d}f_{[p_1p_2]}$, we have the following key observation.

Lemma 2.6. $\mathrm{d}f_{[p_1p_2]}$ *is an isometrical embedding.*

Proof. By Lemma 2.7 below, it suffices to show that $\mathrm{d}f_{[p_1p_2]}$ is a distance nondecreasing map. Note that (2.2) implies that

$$|f_{[p_1p_2]}(p_2')f_{[p_1p_2]}(p_2'')| \geq |p_2'p_2''| \tag{2.8}$$

for all $p_2', p_2'' \in M_2$. It is not hard to see that (2.8) together with (2.6) and (2.7) implies that $\mathrm{d}f_{[p_1p_2]}$ is a distance nondecreasing map. $\square$

Lemma 2.7 ([SSW])**.** *Let N be a complete Alexandrov space with curvature ≥ 1 (especially a complete Riemannian manifold with $\sec_N \geq 1$). If $f : \mathbb{S}^k \to N$ is a distance nondecreasing map, then f is an isometrical embedding.*

Note that Lemma 2.6 implies that there is an $\mathbb{S}^{n_2}$ passing $\uparrow_{p_1}^{p_2}$ in $\mathbb{S}_{p_1}^{n_2+1}$ such that

$$\Sigma_{\uparrow_{p_1}^{p_2}}\mathbb{S}^{n_2} = \mathrm{d}f_{[p_1p_2]}(\Sigma_{p_2}M_2).$$

For convenience, we let $\mathbb{S}^{n_2}_{[p_1p_2]}$ denote this $\mathbb{S}^{n_2}$ (similarly, we have the corresponding $\mathbb{S}^{n_1}_{[p_2p_1]}$ ($\subset (\Sigma_{p_2}M_2)^{=\frac{\pi}{2}} = \mathbb{S}^{n_1+1}$) if $n_1 > 0$). By the definition of $f_{[p_1p_2]}$ (together with Lemma 2.4), it is not hard to see that

$$f_{[p_1p_2]}(M_2) \subseteq \mathbb{S}^{n_2}_{[p_1p_2]}.$$

Remark 2.8. By Lemma 2.6, $\mathrm{d}f_{[p_1p_2]}$ can be generalized naturally to an isometry

$$\mathrm{d}f_{[p_1p_2]} : T_{p_2}M_2 \to T_{\uparrow_{p_1}^{p_2}}\mathbb{S}^{n_2}_{[p_1p_2]}.$$

Then it is easy to see that

$$f_{[p_1p_2]}|_{B_{M_2}(p_2,r_0)} = \exp_{\uparrow_{p_1}^{p_2}} \circ \mathrm{d}f_{[p_1p_2]} \circ \left(\exp_{p_2}|_{B_{T_{p_2}M_2}(O,r_0)}\right)^{-1},$$

where r_0 is the injective radius of M_2 (and O is the original point of $T_{p_2}M_2$), and $\exp_{\uparrow_{p_1}^{p_2}}$ and $\exp_{p_2}$ are the exponential maps of $\mathbb{S}^{n_2}_{[p_1p_2]}$ and M_2 at $\uparrow_{p_1}^{p_2}$ and p_2 respectively. (In the paper, we denote by $B_A(p,r)$ the open r-ball in A with the center p.)

Note that for any $[p_2p_2'] \subset B_{M_2}(p_2, r_0)$, $f_{[p_1p_2]}([p_2p_2'])$ is a minimal geodesic in $\mathbb{S}^{n_2}_{[p_1p_2]}$ (see (2.7)), i.e., $f_{[p_1p_2]}([p_2p_2'])$ lies in a great circle $\mathfrak{S}^1 \subseteq \mathbb{S}^{n_2}_{[p_1p_2]}$. Let $f_{[p_1p_2]}(p_2')$ be the direction of $[p_1p_2']$ (from p_1 to p_2'). We can also consider the map $f_{[p_1p_2']} : M_2 \to \mathbb{S}_{p_1}^{n_2+1}$. Then it is not hard to conclude that:

Lemma 2.9. *$\mathfrak{S}^1 \subset f_{p_1}(M_2)$. And $f_{p_1}^{-1}(\mathfrak{S}^1)$ is a closed geodesic containing $[p_2p_2']$, and $f_{p_1}^{-1}|_{\mathfrak{S}^1}$ is a locally isometrical map.*

Lemma 2.9 has the following almost immediate corollary.

Corollary 2.10. (2.10.1) *For any $[p_1p_2]$ with $p_i \in M_i$, we have that*

$$\mathbb{S}^{n_2}_{[p_1p_2]} \subseteq f_{p_1}(M_2).$$

(2.10.2) *Each minimal geodesic on M_2 lies in a closed geodesic whose length divides 2π.*

3 Proof of The Main Theorem

In this section, all M_i $(i = 1, 2)$ and M are also the manifolds in the Main Theorem, and we assume that

$$n_2 > 0.$$

According to Corollary 2.5, we can divide the whole proof into two parts: one is on $\lambda_{p_1p_2} \equiv h < +\infty$, and the other is on $\lambda_{p_1p_2} = +\infty$ for all $p_i \in M_i$. Hence, the Main Theorem follows from Lemma 3.3 and 3.14 below (in the proof of Lemma 3.14, Lemma 3.10 plays the most important role).

Part a. On $\lambda_{p_1p_2} \equiv h < +\infty$ for all $p_i \in M_i$.

We first give an observation on the condition "$\lambda_{p_1p_2} \equiv h < +\infty$".

Proposition 3.1. $\lambda_{p_1p_2} \equiv h < +\infty$ *for all* $p_i \in M_i$ *if and only if* $n_2 = 1$ *or* $\sec_{M_2} \equiv 1$.

In its proof, we will use the classical Frankel's Theorem.

Theorem 3.2 ([Fr])**.** *Let N^n be a closed positively curved manifold, and let $N_i^{n_i}$ $(i = 1, 2)$ be complete totally geodesic submanifolds in N. If $n_1 + n_2 \geq n$, then $N_1 \cap N_2 \neq \emptyset$.*

Proof of Proposition 3.1.

If $\lambda_{p_1p_2} \equiv h < +\infty$ for all $p_i \in M_i$, the by (3.4.1) below, for any $[p_1p_2]$ with $p_i \in M_i$, there is a neighborhood $U \subset M_2$ of p_2 such that $f_{[p_1p_2]}|_U : U \to \mathbb{S}^{n_2}_{[p_1p_2]}$ is an isometrical embedding, which implies that $\sec_{M_2} \equiv 1$ if $n_2 > 1$.

Next, it suffices to show that $\lambda_{p_1p_2} < +\infty$ (see Corollary 2.5) by assuming that $n_2 = 1$ or $\sec_{M_2} \equiv 1$. We fix a $[p_1p_2]$ with $p_i \in M_i$, and consider $\mathbb{S}^{n_2}_{[p_1p_2]}$. For any $\xi \in \mathbb{S}^{n_2}_{[p_1p_2]}$, there is a $[p_1p_2']$ with $p_2' \in M_2$ such that $\xi = \uparrow_{p_1}^{p_2'}$ (see (2.10.1)). **Claim:**

$$\mathbb{S}^{n_2}_{[p_1p_2]} = \mathbb{S}^{n_2}_{[p_1p_2']}.$$

If $n_2 = 1$, then $f_{p_1}^{-1}|_{\mathbb{S}^1_{[p_1p_2]}} : \mathbb{S}^1_{[p_1p_2]} \to M_2$ is a locally isometrical map (by Lemma 2.9), which implies the claim. If $\sec_{M_2} \equiv 1$, then by Remark 2.8 we have that $f_{[p_1p_2]}|_{B_{M_2}(p_2, \frac{r_0}{2})} : B_{M_2}(p_2, \frac{r_0}{2}) \to B_{\mathbb{S}^{n_2}_{[p_1p_2]}}(\uparrow_{p_1}^{p_2}, \frac{r_0}{2})$ is an isometry (where r_0 is the injective radius of M_2). It then follows that for any $\eta \in B_{\mathbb{S}^{n_2}_{[p_1p_2]}}(\uparrow_{p_1}^{p_2}, \frac{r_0}{2})$ we have that $\mathbb{S}^{n_2}_{[p_1p_2]} = \mathbb{S}^{n_2}_{[p_1p_2'']}$, where $[p_1p_2'']$ with $p_2'' \in M_2$ satisfies $\uparrow_{p_1}^{p_2''} = \eta$. Then it is not hard to see that the claim follows. On the other hand, by Theorem 3.2 we have that

$$\mathbb{S}^{n_2}_{[p_1p_2]} \cap \mathbb{S}^{n_2}_{[p_1p_2]'} \neq \emptyset$$

for any other minimal geodesic $[p_1p_2]'$ between p_1 and p_2. It then follows that

$$\mathbb{S}^{n_2}_{[p_1p_2]} = \mathbb{S}^{n_2}_{[p_1p_2]'}.$$

Hence, $f_{p_1}(M_2) = \mathbb{S}^{n_2}_{[p_1p_2]}$, and $f_{p_1}^{-1} : \mathbb{S}^{n_2}_{[p_1p_2]} \to M_2$ is a Riemannian covering map, which implies that $\lambda_{p_1p_2} < +\infty$. $\square$

Now we classify M_i and M under the condition "$\lambda_{p_1p_2} \equiv h < +\infty$".

Lemma 3.3. *If $\lambda_{p_1p_2} \equiv h < +\infty$ for all $p_i \in M_i$, then* (i) *when $n \geq 5$, $M_i \overset{\text{iso}}{\cong} \mathbb{S}^{n_i}/\mathbb{Z}_h$ and $M \overset{\text{iso}}{\cong} \mathbb{S}^n/\mathbb{Z}_h$, and $h \geq 3$ implies that $n_1 = 0$;* (ii) *when $n = 4$, $M_2 \overset{\text{iso}}{\cong} \mathbb{RP}^2$ and $M \overset{\text{iso}}{\cong} \mathbb{RP}^4$ (resp. $M_2 \overset{\text{iso}}{\cong} \mathbb{S}^2$ and $M \overset{\text{iso}}{\cong} \mathbb{S}^4$, or $M_2 \overset{\text{iso}}{\cong} \mathbb{RP}^2$) if M is not simply connected (resp. M is simply connected);* (iii) *when $n = 3$, $M_2 \overset{\text{iso}}{\cong} \mathbb{S}^1$ and $M \overset{\text{iso}}{\cong} \mathbb{S}^3$, or $M_2 \overset{\text{iso}}{\cong} \mathbb{S}^1/\mathbb{Z}_h$ with $h \geq 2$.*

In the proof of Lemma 3.3, we will use the following technical lemmas. (For the convenience of readers, we will give a brief proof for (3.4.1) in Appendix.)

Lemma 3.4 ([RW]). *If $\lambda_{p_1p_2} \equiv h < +\infty$ for all $p_i \in M_i$, then for any $[p_1p_2]$*
(3.4.1) *there is a neighborhood $U \subset M_2$ of p_2 such that $f_{[p_1p_2]}|_U$ is an isometry;*
(3.4.2) *there are neighborhoods $U_i \subset M_i$ of p_i such that $U_1 * U_2$* [4] *can be embedded isometrically into M around $[p_1p_2]$.*

Lemma 3.5. *Let N^m be a complete Riemannian manifold with $\sec_N \geq 1$, and let L^l be a complete totally geodesic submanifold in N with $l \geq \frac{m}{2}$. Assume that $\sec_L \equiv 1$.*
(3.5.1) *If $l > \frac{m}{2}$, then we have that $\sec_N \equiv 1$; and if $m - l = 2$ (resp. $m - l$ is odd) additionally, then $\pi_1(N) = \pi_1(L) = \mathbb{Z}_k$ (resp. $\mathbb{Z}_2$ or 1) for some k;*
(3.5.2) *If $l = \frac{m}{2}$, and if $L \overset{\text{iso}}{\cong} \mathbb{RP}^l$ and N is not simply connected additionally, then we have that $N \overset{\text{iso}}{\cong} \mathbb{RP}^m$ with the canonical metric.*

In the proof of Lemma 3.5, we will use the following connectedness theorem.

Theorem 3.6 ([Wi2]). *Let N^m be a closed positively curved manifold, and let L^l be a complete totally geodesic submanifold in N with $l \geq \frac{m}{2}$. Then $L \hookrightarrow N$ is $(2l - m + 1)$-connected.*

Proof of Lemma 3.5.

Let $\pi : \tilde{N} \to N$ be the Riemannian covering map, and let $\tilde{L} = \pi^{-1}(L)$ which is complete and totally geodesic in $\tilde{N}$. By Theorem 3.2, $\tilde{L}$ is connected because $l \geq \frac{m}{2}$.

(3.5.1) Since $l > \frac{m}{2}$, both $L \hookrightarrow N$ and $\tilde{L} \hookrightarrow \tilde{N}$ are at least 2-connected (Theorem 3.6). This implies that $\pi_1(N) = \pi_1(L)$ and $\tilde{L}$ is simply connected. It then follows that $\tilde{L} \overset{\text{iso}}{\cong} \mathbb{S}^l$ (note that $\sec_L \equiv 1$). By the Maximal Diameter Theorem, it has to hold that $\tilde{N} \overset{\text{iso}}{\cong} \mathbb{S}^m$, i.e. $\sec_N \equiv 1$. And it is easy to see that $\pi_1(N) = \pi_1(L) = \mathbb{Z}_2$ or 1 if $m - l$ is odd. Now we assume that $m - l = 2$. Note that there is a great circle $\mathbb{S}^1$ such that $\tilde{N} = \tilde{L} * \mathbb{S}^1$ (i.e. $\mathbb{S}^m = \mathbb{S}^l * \mathbb{S}^1$). On the other hand, $\pi_1(N)$ ($= \pi_1(L)$) acts on $\tilde{N}$ freely by isometries. Moreover, $\pi_1(N)$ preserves $\tilde{L}$, and thus it also preserves the $\mathbb{S}^1$. It follows that $\pi_1(N) = \pi_1(L) = \mathbb{Z}_k$ for some k.

(3.5.2) Since $m = 2l$, we know that $\pi_1(N) = \mathbb{Z}_2$. On the other hand, since $\tilde{L} = \pi^{-1}(L)$ (which is connected) and $L \overset{\text{iso}}{\cong} \mathbb{RP}^l$, it has to hold that $\tilde{L} \overset{\text{iso}}{\cong} \mathbb{S}^l$. Similarly, by the Maximal Diameter Theorem we get that $\tilde{N} \overset{\text{iso}}{\cong} \mathbb{S}^m$, and so $N \overset{\text{iso}}{\cong} \mathbb{RP}^m$. $\square$

[4] Refer to A.3 in Appendix for the metric of $U_1 * U_2$.

Now we give the proof of Lemma 3.3.

Proof of Lemma 3.3.

From the proof of Proposition 3.1, for a fixed $[p_1p_2]$ with $p_i \in M_i$, $f_{p_1}^{-1} : \mathbb{S}^{n_2}_{[p_1p_2]} \to M_2$ is a Riemannian covering map, and $\#(\pi_1(M_2)) = h$ when $n_2 \geq 2$. Next, we will divide the proof into the following two cases.

Case 1: $n_1 = 0$. In this case, $n_2 = n - 2$. If $n \geq 5$, then by (3.5.1) we have that $M_2 \overset{\text{iso}}{\cong} \mathbb{S}^{n_2}/\mathbb{Z}_h$, and $M \overset{\text{iso}}{\cong} \mathbb{S}^n/\mathbb{Z}_h$ (note that $n_2 > \frac{n}{2}$ and $\#(\pi_1(M_2)) = h$). If $n = 4$, then $M_2 \overset{\text{iso}}{\cong} \mathbb{S}^2$ or $\mathbb{RP}^2$ (note that $n_2 = 2$), and thus respectively, $M \overset{\text{iso}}{\cong} \mathbb{S}^4$ by the Maximum Diameter Theorem or $M \overset{\text{iso}}{\cong} \mathbb{RP}^4$ by (3.5.2) if M is not simply connected. If $n = 3$, then $M_2 \overset{\text{iso}}{\cong} \mathbb{S}^1/\mathbb{Z}_h$ (note that $n_2 = 1$), and thus $M \overset{\text{iso}}{\cong} \mathbb{S}^3$ by the Maximum Diameter Theorem if $M_2 \overset{\text{iso}}{\cong} \mathbb{S}^1$.

Case 2: $n_1 > 0$. In this case, for any $p_i \in M_i$, we have proved that $f_{p_1}(M_2) = \mathbb{S}^{n_2}$ and $f_{p_2}(M_1) = \mathbb{S}^{n_1}$, and both $f_{p_1}^{-1} : \mathbb{S}^{n_2} \to M_2$ and $f_{p_2}^{-1} : \mathbb{S}^{n_1} \to M_1$ are Riemannian covering maps. Together with (3.4.2), this implies that the set

$$N \triangleq \{p \in M |\ p \text{ belongs to some } [p_1p_2] \text{ with } p_i \in M_i\}$$

is a complete totally geodesic $(n-1)$-dimensional submanifold in M (Hint: It follows from Lemma 2.3 that, for any $p_i \in M_i$, say p_1, $\Sigma_{p_1}N = f_{p_1}(M_2)*\Sigma_{p_1}M_1 = \mathbb{S}^{n_2}*\mathbb{S}^{n_1-1} = \mathbb{S}^{n-2}$). Hence, by Corollary 0.2, we know that $\sec_N \equiv 1$ (note that M_1 and M_2 are totally geodesic in N). It then follows from (3.5.1) that M is isometric to $\mathbb{S}^n$ or $\mathbb{RP}^n$ (and so M_i is isometric to $\mathbb{S}^{n_i}$ or $\mathbb{RP}^{n_i}$ respectively). $\square$

Remark 3.7. Why cannot we prove that $M \overset{\text{iso}}{\cong} \mathbb{RP}^4$ (resp. $M \overset{\text{iso}}{\cong} \mathbb{S}^3/\mathbb{Z}_h$ with $h \geq 2$) when $n = 4$ and $M_2 \overset{\text{iso}}{\cong} \mathbb{RP}^2$ (resp. $n = 3$ and $M_2 \overset{\text{iso}}{\cong} \mathbb{S}^1/\mathbb{Z}_h$) by a similar argument to the above proof for $n_1 > 0$? Note that in such two cases, $n_1 = 0$, i.e. $M_1 = \{p_1\}$. Due to the similarity, we only give an explanation for the case where $n = 4$. Note that $\lambda_{p_1p_2} \equiv 2$ in this case, i.e., there are only two minimal geodesics $[p_1p_2]_j$ $(j = 1, 2)$ between p_1 and any $p_2 \in M_2$. Since $f_{p_1}^{-1} : \mathbb{S}^2_{[p_1p_2]_j} \to M_2$ is a Riemannian covering map, $[p_1p_2]_1$ and $[p_1p_2]_2$ form an angle equal to π at p_1. However, we cannot judge whether they form an angle equal to π at p_2 or not, so that we cannot judge whether $N \triangleq \{p \in M |\ p \text{ belongs to some } [p_1p_2] \text{ with } p_2 \in M_2\}$ is totally geodesic in M or not.

Part b. On $\lambda_{p_1p_2} = +\infty$ for all $p_i \in M_i$.

Lemma 3.8. *If $\lambda_{p_1p_2} = +\infty$ for all $p_i \in M_i$, then both n_1 and n_2 are even.*

Proof. By Proposition 3.1, it suffices to derive a contradiction by assuming that $n_2 = 2m + 1$ with $m > 0$. We still fix an arbitrary $[p_1p_2]$ with $p_i \in M_i$ at first. And, in this proof, we always let $\tilde{q}$ denote $f_{[p_1p_2]}(q)$ for any $q \in B_{M_2}(p_2, \frac{r_0}{2})$, where r_0 is the injective radius of M_2.

Claim 1: *There is an $\mathbb{S}^{m+1} \subset \mathbb{S}^{2m+1}_{[p_1p_2]}$ such that $f_{p_1}^{-1}|_U$ is an isometry for some convex domain U in the $\mathbb{S}^{m+1}$*. We will find such an $\mathbb{S}^{m+1}$ through the following steps.

Step 1. We select an arbitrary $\tilde{p}_2^1 \in B_{\mathbb{S}^{2m+1}_{[p_1p_2]}}(\tilde{p}_2, \frac{r_0}{2}) \setminus \{\tilde{p}_2\}$. By the definition of $f_{[p_1p_2]}$, there is a $[p_1p_2^1]$ with $\uparrow_{p_1}^{p_2^1}= \tilde{p}_2^1$ such that $f_{p_1}^{-1}|_{[\tilde{p}_2\tilde{p}_2^1]} : [\tilde{p}_2\tilde{p}_2^1] \to [p_2p_2^1]$ is an isometry (see (2.7)). For convenience, we denote by $\mathbb{S}^1_\bullet$ the great circle including $[\tilde{p}_2\tilde{p}_2^1]$ in $S^{2m+1}_{[p_1p_2]}$.

We also consider $\mathbb{S}^{2m+1}_{[p_1p_2^1]}$, and observe that

$$[\tilde{p}_2\tilde{p}_2^1] \subset \mathbb{S}^{2m+1}_{[p_1p_2]} \cap \mathbb{S}^{2m+1}_{[p_1p_2^1]}$$

and

$$\mathbb{S}^{2m+1}_{[p_1p_2]} \cap \mathbb{S}^{2m+1}_{[p_1p_2^1]} = \mathbb{S}^{k_1} \text{ (denoted by } \mathbb{S}^{k_1}_{[p_1p_2]}) \text{ with } k_1 \geq 2m \tag{3.1}$$

(note that $\mathbb{S}^{2m+1}_{[p_1p_2]}, \mathbb{S}^{2m+1}_{[p_1p_2^1]} \subset \mathbb{S}^{2m+2}_{p_1}$). Note that

$$f_{p_1}^{-1}(B_{\mathbb{S}^{k_1}_{[p_1p_2]}}(\tilde{p}_2, \frac{r_0}{2})) \subset B_{M_2}(p_2, \frac{r_0}{2}) \cap B_{M_2}(p_2^1, r_0),$$

and thus, for any $\tilde{p}_2' \in B_{\mathbb{S}^{k_1}_{[p_1p_2]}}(\tilde{p}_2, \frac{r_0}{2})$, we have that

$$|\tilde{p}_2\tilde{p}_2^1| = |p_2p_2^1|, |\tilde{p}_2\tilde{p}_2'| = |p_2p_2'|, |\tilde{p}_2^1\tilde{p}_2'| = |p_2^1p_2'|.$$

Moreover, note that $\angle p_2'p_2p_2^1 = \angle \tilde{p}_2'\tilde{p}_2\tilde{p}_2^1$ (Lemma 2.6 and Remark 2.8). It then follows from (iii) of Theorem 1.1 that

$$\textit{the triangle } \triangle p_2p_2^1p_2' \textit{ bounds a convex spherical surface in } M_2, \tag{3.2}$$

where the triangle $\triangle p_2p_2^1p_2'$ is formed by $[p_2p_2^1]$, $[p_2p_2']$ and $[p_2^1p_2']$ (note that there is a unique minimal geodesic between any two points in $B_{M_2}(p_2, \frac{r_0}{2})$).

Step 2. We select $\tilde{p}_2^2 \in B_{\mathbb{S}^{k_1}_{[p_1p_2]}}(\tilde{p}_2, \frac{r_0}{2}) \setminus \mathbb{S}^1_\bullet$, and let $[p_1p_2^2]$ be the minimal geodesic such that $\tilde{p}_2^2 =\uparrow_{p_1}^{p_2^2}$. And we let $\mathbb{S}^2_\bullet$ be the unit sphere $\mathbb{S}^2 \subset \mathbb{S}^{k_1}_{[p_1p_2]}$ including $\tilde{p}_2, \tilde{p}_2^1, \tilde{p}_2^2$, and let D be the convex domain in $\mathbb{S}^2_\bullet$ bounded by $\triangle\tilde{p}_2\tilde{p}_2^1\tilde{p}_2^2$. By (3.2), it is easy to see that $f_{p_1}^{-1}|_D$ is an isometry.

Similarly, we consider

$$\mathbb{S}^{k_1}_{[p_1p_2]} \cap \mathbb{S}^{2m+1}_{[p_1p_2^2]} = \mathbb{S}^{k_2} \text{ (denoted by } \mathbb{S}^{k_2}_{[p_1p_2]}) \text{ with } k_2 \geq 2m-1;$$

and, for any $\tilde{p}_2' \in B_{\mathbb{S}^{k_2}_{[p_1p_2]}}(\tilde{p}_2, \frac{r_0}{2})$, by (iii) of Theorem 1.1, we can derive that

$$\{p_2', p_2, p_2^1, p_2^2\} \text{ as vertices determines a convex spherical tetrahedron in } M_2. \tag{3.3}$$

$\cdots$

Step $m+1$. We select $\tilde{p}_2^{m+1} \in B_{\mathbb{S}^{k_m}_{[p_1p_2]}}(\tilde{p}_2, \frac{r_0}{2}) \setminus \mathbb{S}^m_\bullet$. Let $\mathbb{S}^{m+1}_\bullet$ be the unit sphere $\mathbb{S}^{m+1} \subset \mathbb{S}^{k_m}_{[p_1p_2]}$ including $\tilde{p}_2, \tilde{p}_2^1, \cdots, \tilde{p}_2^{m+1}$ which as vertices determines a convex domain U in $\mathbb{S}^{m+1}_\bullet$. Similarly, by the corresponding property similar to (3.2) and (3.3) in the

m-th step, we have that $f_{p_1}^{-1}|_U$ is an isometry. That is, $\mathbb{S}_{\bullet}^{m+1}$ is just the wanted sphere in Claim 1.

In fact, Claim 1 has the following strengthened version.

Claim 2: *For any $\tilde{p}_2' \in \mathbb{S}_{\bullet}^{m+1}$, $f_{p_1}^{-1}|_{B_{\mathbb{S}_{\bullet}^{m+1}}(\tilde{p}_2', \frac{r_0}{2})}$ is an isometry.* We first select a $\tilde{p}_{2,0}'$ in the interior part of $U \subset \mathbb{S}_{\bullet}^{m+1}$. Let $p_{2,0}' = f_{p_1}^{-1}(\tilde{p}_{2,0}')$ and $[p_1 p_{2,0}']$ be the minimal geodesic such that $\tilde{p}_{2,0}' = \uparrow_{p_1}^{p_{2,0}'}$. Since $f_{p_1}^{-1}|_U$ is an isometry, it is easy to see that $\mathbb{S}_{\bullet}^{m+1} \subset \mathbb{S}_{[p_1 p_{2,0}']}^{2m+1} \cap \mathbb{S}_{[p_1 p_2'']}^{2m+1}$, where $[p_1 p_2'']$ is the minimal geodesic such that $\uparrow_{p_1}^{p_2''}$ belongs to U and is close to $\tilde{p}_{2,0}'$. By the arguments to get (3.2) we can conclude that $f_{p_1}^{-1}|_{B_{\mathbb{S}_{\bullet}^{m+1}}(\tilde{p}_{2,0}', \frac{r_0}{2})}$ is an isometry. Then by replacing U with $B_{\mathbb{S}_{\bullet}^{m+1}}(\tilde{p}_{2,0}', \frac{r_0}{2})$, it is not hard to see that $f_{p_1}^{-1}|_{B_{\mathbb{S}_{\bullet}^{m+1}}(\tilde{p}_2', \frac{r_0}{2})}$ is an isometry for any $\tilde{p}_2' \in \mathbb{S}_{\bullet}^{m+1}$.

Inspired by the proof of Claim 2, we have the following observation.

Claim 3: *For any small $\varepsilon > 0$, there is another minimal geodesic $[p_1 p_2]'$ between p_1 and p_2 such that $|\uparrow_{p_1}^{p_2} (\uparrow_{p_1}^{p_2})'| < \varepsilon$.* In fact, if this is not true, then based on (2.5) we can use a similar proof of (3.4.1) (ref. A.2 in Appendix) to prove that there is a neighborhood $V \subset M_2$ of p_2 such that $f_{[p_1 p_2]}|_V$ is an isometry, so is $f_{p_1}^{-1}|_{f_{[p_1 p_2]}(V)}$. Note that $f_{[p_1 p_2]}(V)$ is an open subset in $\mathbb{S}_{[p_1 p_2]}^{2m+1}$, so the proof of Claim 2 implies that $f_{p_1}^{-1}|_{B_{\mathbb{S}_{[p_1 p_2]}^{2m+1}}(\tilde{p}_2', \frac{r_0}{2})}$ is an isometry for any $\tilde{p}_2' \in \mathbb{S}_{[p_1 p_2]}^{2m+1}$. This implies that $f_{p_1}^{-1}|_{\mathbb{S}_{[p_1 p_2]}^{2m+1}}$ is a Riemannian covering map, which contradicts Proposition 3.1.

Now, we will complete the whole proof of the lemma based on Claims 1-3. We still consider the above $\mathbb{S}_{\bullet}^{m+1} \subset \mathbb{S}_{[p_1 p_2]}^{2m+1}$ (with $\uparrow_{p_1}^{p_2} \in \mathbb{S}_{\bullet}^{m+1}$). Let $[p_1 p_2]'$ be another minimal geodesic between p_1 and p_2 with $(\uparrow_{p_1}^{p_2})'$ being sufficiently close to $\uparrow_{p_1}^{p_2}$. We consider the natural isometry (ref. Remark 2.8)

$$h \triangleq \exp_{(\uparrow_{p_1}^{p_2})'} \circ \mathrm{d}f_{[p_1 p_2]'} \circ \mathrm{d}f_{[p_1 p_2]}^{-1} \circ \exp_{\uparrow_{p_1}^{p_2}}^{-1} : \mathbb{S}_{[p_1 p_2]}^{2m+1} \to \mathbb{S}_{[p_1 p_2]'}^{2m+1}.$$

Duo to (2.5) and that $(\uparrow_{p_1}^{p_2})'$ is sufficiently close to $\uparrow_{p_1}^{p_2}$, it is easy to see that

$$|h(\tilde{p}_2')\tilde{p}_2'| \ll \frac{r_0}{2} \text{ (in } \mathbb{S}_{p_1}^{2m+2}) \text{ for any } \tilde{p}_2' \in \mathbb{S}_{[p_1 p_2]}^{2m+1}.$$

On the other hand, it is not hard to see that the unit sphere $h(\mathbb{S}_{\bullet}^{m+1})$ (containing $(\uparrow_{p_1}^{p_2})'$) satisfies that $f_{p_1}^{-1}|_{B_{h(\mathbb{S}_{\bullet}^{m+1})}(\tilde{p}_2'', \frac{r_0}{2})}$ is an isometry for any $\tilde{p}_2'' \in h(\mathbb{S}_{\bullet}^{m+1})$. By Theorem 3.2, we have that $\mathbb{S}_{\bullet}^{m+1} \cap h(\mathbb{S}_{\bullet}^{m+1}) \neq \emptyset$ in $\mathbb{S}_{p_1}^{2m+2}$. Select a $\tilde{q}$ in $\mathbb{S}_{\bullet}^{m+1} \cap h(\mathbb{S}_{\bullet}^{m+1})$, and let $q = f_{p_1}^{-1}(\tilde{q})$. Since $f_{p_1}^{-1}|_{B_{h(\mathbb{S}_{\bullet}^{m+1})}(\tilde{q}, \frac{r_0}{2})}$ is an isometry and $|h(\tilde{q})\tilde{q}| < \frac{r_0}{2}$, it has to hold that $h(\tilde{q}) = \tilde{q}$ (note that $\tilde{q}, h(\tilde{q}) \in f_{p_1}(q)$), which implies that $f_{p_1}^{-1}([\uparrow_{p_1}^{p_2} \tilde{q}])$ and $f_{p_1}^{-1}([(\uparrow_{p_1}^{p_2})'\tilde{q}])$ are the same geodesic between p_2 and q in M_2. This contradicts Lemma 2.6 once $f_{[p_1 q]}$ is considered, where $[p_1 q]$ is the minimal geodesic such that $\uparrow_{p_1}^{q} = \tilde{q}$. □

Remark 3.9. If $n_2 = 2m+2$ in the above proof, then $\mathbb{S}_{\bullet}^{m+1} \cap h(\mathbb{S}_{\bullet}^{m+1})$ can be empty in $\mathbb{S}_{p_1}^{2m+3}$ (and we can not find an $\mathbb{S}^{m+2} \subset \mathbb{S}_{[p_1 p_2]}^{2m+2}$ such that $f_{p_1}^{-1}|_{\mathbb{S}^{m+2}}$ is a local isometry). In fact, if $M \overset{\text{iso}}{\cong} \mathbb{CP}^{\frac{n}{2}}$ and $M_2 \overset{\text{iso}}{\cong} \mathbb{CP}^{m+1}$, then $f_{p_1}^{-1}(\mathbb{S}_{\bullet}^{m+1})$ is a complete totally geodesic submanifold (in M_2) which is isometric to $\mathbb{RP}^{m+1}$ with the canonical metric.

By Lemma 3.8, we can assume that $n_2 = 2m > 0$.

Lemma 3.10. *If $\lambda_{p_1p_2} = +\infty$ for all $p_i \in M_i$, then M_2 is isometric to $\mathbb{CP}^m$ or $\mathbb{CP}^m/\mathbb{Z}_2$ with the canonical metric, and that $M_2 \overset{\text{iso}}{\cong} \mathbb{CP}^m/\mathbb{Z}_2$ occurs only when m is odd.*

In order to prove Lemma 3.10, we first give a key observation.

Lemma 3.11. *For any $\epsilon > 0$, there is a $\delta > 0$ such that if $|(\uparrow_{p_1}^{p_2})_1(\uparrow_{p_1}^{p_2})_2| < \delta$ for arbitrary two minimal geodesics $[p_1p_2]_1$ and $[p_1p_2]_2$ between $p_1 \in M_1$ and $p_2 \in M_2$, then*

$$\left| |\uparrow_{\tilde{p}_2^1}^{\tilde{p}_2^2} \xi| - \frac{\pi}{2} \right| < \epsilon$$

for any $\xi \in \Sigma_{\tilde{p}_2^1}\mathbb{S}^{2m}_{[p_1p_2]_1}$, where $\tilde{p}_2^j$ denotes $(\uparrow_{p_1}^{p_2})_j$.

Proof. On $\mathbb{S}^{2m+1}_{p_1}$, for any $p_2' \in B_{M_2}(p_2, r_0)$ (where r_0 is the injective radius of M_2),

$$f_{[p_1p_2]_1}(p_2') \in \mathbb{S}^{2m}_{[p_1p_2]_1} \text{ and } |\tilde{p}_2^1 f_{[p_1p_2]_1}(p_2')| = |p_2p_2'|,$$

and by (ii) of Theorem 1.1

$$|\tilde{p}_2^2 f_{[p_1p_2]_1}(p_2')| \geq |p_2p_2'|.$$

It then is easy to see that the lemma follows from the first variation formula. $\square$

Proof of Lemma 3.10.

We still fix an arbitrary $[p_1p_2]$ with $p_i \in M_i$ at first, and consider $f_{p_1}, f_{[p_1p_2]}, \mathbb{S}^{2m+1}_{p_1}, \mathbb{S}^{2m}_{[p_1p_2]}$ and so on. By (2.7), for any $[p_2p_2'] \subset B_{M_2}(p_2, \frac{r_0}{2})$, there is a $[p_1p_2']$ such that $f_{[p_1p_2]}|_{[p_2p_2']}$: $[p_2p_2'] \to [\uparrow_{p_1}^{p_2}\uparrow_{p_1}^{p_2'}]$ is an isometry. Similar to the proof of Lemma 3.8, we have that

$$[\uparrow_{p_1}^{p_2}\uparrow_{p_1}^{p_2'}] \subset \mathbb{S}^{2m}_{[p_1p_2]} \cap \mathbb{S}^{2m}_{[p_1p_2']},$$

and we will consider (similar to (3.1))

$$\mathbb{S}^{k_1}_{[p_1p_2],[p_2p_2']} \triangleq \mathbb{S}^{2m}_{[p_1p_2]} \cap \mathbb{S}^{2m}_{[p_1p_2']} \text{ with } k_1 \geq 2m-1. \tag{3.4}$$

Claim 1: *In fact, we have that $k_1 = 2m-1$.* If $k_1 = 2m$, then similar to Claims 1 and 2 in the proof of Lemma 3.8 we can find an $\mathbb{S}^{m+1} \subset \mathbb{S}^{2m+1}_{p_1}$ such that $f_{p_1}^{-1}|_{\mathbb{S}^{m+1}}$ is a local isometry, and we can obtain a contradiction.

For convenience, we let $\gamma(t)|_{t\in[0,|p_2p_2'|]}$ denote the $[p_2p_2']$ with $\gamma(0) = p_2$ (t is the arc-length parameter), and let $\tilde{\gamma}(t)$ denote $f_{[p_1p_2]}(\gamma(t))$. (In this proof, we also let $\tilde{q}$ denote $f_{[p_1p_2]}(q)$ for any $q \in B_{M_2}(p_2, \frac{r_0}{2})$). Let $[p_1\gamma(t)]$ be the minimal geodesic such that $\uparrow_{p_1}^{\gamma(t)} = \tilde{\gamma}(t)$. We claim that

$$\mathbb{S}^{2m}_{[p_1\gamma(t)]} \cap \mathbb{S}^{2m}_{[p_1\gamma(t')]} = \mathbb{S}^{2m-1}_{[p_1p_2],[p_2p_2']} \text{ for all } t \neq t'. \tag{3.5}$$

We need only to verify it for $m \geq 2$. Note that for any $\mathbb{S}^2 \subset \mathbb{S}^{2m-1}_{[p_1p_2],[p_2p_2']}$ containing $[\tilde{p}_2\tilde{p}_2']$, $f_{p_1}^{-1}|_{B_{\mathbb{S}^2}(\tilde{p}_2,\frac{r_0}{2})}$ is an isometry (which is similar to "$f_{p_1}^{-1}|_{B_{\mathbb{S}^{m+1}_\bullet}(\tilde{p}_2,\frac{r_0}{2})}$ is an isometry" in Claim 2 in the proof of Lemma 3.8, and implies that $f_{p_1}^{-1}(B_{\mathbb{S}^2}(\tilde{p}_2, \frac{r_0}{2}))$ is totally

geodesic in M_2). This implies that $\mathbb{S}^2 \subset \mathbb{S}^{2m}_{[p_1\gamma(t)]} \cap \mathbb{S}^{2m}_{[p_1\gamma(t')]}$, and so (3.5) follows from Claim 1. Moreover, a parallel vector field $X(t)$ along $\gamma(t)$ on $f_{p_1}^{-1}(B_{\mathbb{S}^2}(\tilde{p}_2, \frac{r_0}{2}))$ is also parallel on M_2, and we can naturally define $\mathrm{d}f_{[p_1p_2]}(X(t))$, denoted by $\tilde{X}(t)$, which is parallel along $\tilde{\gamma}(t)$ on $\mathbb{S}^2$ and satisfies $|\tilde{X}(t)| = |X(t)|$. Note that we can select such $2m-2$ parallel and orthogonal (unit) vector fields $X_1(t), \cdots, X_{2m-2}(t)$ along $\gamma(t)$ which are all perpendicular to $\gamma'(t)$, the tangent vector of $\gamma(t)$. And by Lemma 2.6, $\tilde{X}_1(t), \cdots, \tilde{X}_{2m-2}(t)$ are also orthogonal and perpendicular to $\tilde{\gamma}'(t)$.

Now we select a parallel unit vector field $X_{2m-1}(t)$ along $\gamma(t)$ (on M_2) which is perpendicular to $X_1(t), \cdots, X_{2m-2}(t)$ and $\gamma'(t)$.

Claim 2: *We can define* $\mathrm{d}f_{[p_1p_2]}(X_{2m-1}(t))$*, denoted by* $\tilde{X}_{2m-1}(t)$*, which is smooth with respect to* t *and perpendicular to* $\tilde{X}_1(t), \cdots, \tilde{X}_{2m-2}(t)$ *and* $\tilde{\gamma}'(t)$*, and satisfies* $|\tilde{X}_{2m-1}(t)| \geq 1$. Let $\beta_t(s)|_{s\in[0,\epsilon)} \subset B_{M_2}(p_2, \frac{r_0}{2})$ be a geodesic such that $\beta_t(0) = \gamma(t)$ and $\beta_t'(0) = X_{2m-1}(t)$. Due to Remark 2.8, each $\tilde{\beta}_t(s)|_{s\in[0,\epsilon)}$ is a smooth curve with respect to s (but it will not be a geodesic when $t > 0$). It then follows that we can define

$$\mathrm{d}f_{[p_1p_2]}(X_{2m-1}(t)) \triangleq \tilde{\beta}_t'(s)|_{s=0} \text{ (denoted by } \tilde{X}_{2m-1}(t)),$$

which is smooth with respect to t (this is also due to Remark 2.8). On the other hand, since $|\tilde{p}_2\tilde{\beta}_t(s)| = |p_2\beta_t(s)|$ for all $s \in [0,\epsilon)$ and $|\uparrow_{\gamma(t)}^{p_2}\uparrow_{\gamma(t)}^{\beta_t(s)}| = \frac{\pi}{2}$, we have that (by the first variation formula)

$$\lim_{s\to 0}\left|\uparrow_{\tilde{\gamma}(t)}^{\tilde{p}_2}\uparrow_{\tilde{\gamma}(t)}^{\tilde{\beta}_t(s)}\right| = \frac{\pi}{2} \text{ (i.e. } |\tilde{\gamma}'(t)\tilde{X}_{2m-1}(t)| = \frac{\pi}{2})$$

(due to Remark 2.8, one can also get this by Gauss's Lemma). Next we will show that

$$\lim_{s\to 0}\left|\tilde{X}_j(t)\uparrow_{\tilde{\gamma}(t)}^{\tilde{\beta}_t(s)}\right| = \frac{\pi}{2} \text{ (i.e. } |\tilde{X}_j(t)\tilde{X}_{2m-1}(t)| = \frac{\pi}{2}) \text{ for any } 1 \leq j \leq 2m-2. \tag{3.6}$$

Let $[p_{2j}^tq_{2j}^t] \subset B_{M_2}(p_2, \frac{r_0}{2})$ with $\gamma(t) \in [p_{2j}^tq_{2j}^t]^\circ$ be a geodesic such that $X_j(t) = \uparrow_{\gamma(t)}^{p_{2j}^t}$. From the choice of $X_j(t)$, we know that $f_{[p_1p_2]}([p_{2j}^tq_{2j}^t]) = [\tilde{p}_{2j}^t\tilde{q}_{2j}^t] \subset \mathbb{S}^{2m-1}_{[p_1p_2],[p_2p_2']}$ with $|p_{2j}^tq_{2j}^t| = |\tilde{p}_{2j}^t\tilde{q}_{2j}^t|$, and that $\tilde{X}_j(t) = \uparrow_{\tilde{\gamma}(t)}^{\tilde{p}_{2j}^t}$. Note that $|\tilde{p}_{2j}^t\tilde{\gamma}(t)| = |p_{2j}^t\gamma(t)|, |\tilde{p}_{2j}^t\tilde{\beta}_t(s)| \geq |p_{2j}^t\beta_t(s)|$ (by (2.8)) and $|\uparrow_{\gamma(t)}^{p_{2j}^t}\uparrow_{\gamma(t)}^{\beta_t(s)}| = \frac{\pi}{2}$. Then by the first variation formula we have that $\lim\limits_{s\to 0}|\uparrow_{\tilde{\gamma}(t)}^{\tilde{p}_{2j}^t}\uparrow_{\tilde{\gamma}(t)}^{\tilde{\beta}_t(s)}| \geq \frac{\pi}{2}$, and similarly $\lim\limits_{s\to 0}|\uparrow_{\tilde{\gamma}(t)}^{\tilde{q}_{2j}^t}\uparrow_{\tilde{\gamma}(t)}^{\tilde{\beta}_t(s)}| \geq \frac{\pi}{2}$. Hence, (3.6) follows because $|\uparrow_{\tilde{\gamma}(t)}^{\tilde{p}_{2j}^t}\uparrow_{\tilde{\gamma}(t)}^{\tilde{q}_{2j}^t}| = \pi$. On the other hand, note that $|\tilde{\beta}_t(s)\tilde{\gamma}(t)| \geq |\beta_t(s)\gamma(t)|$ (by (2.8)), and so $|\tilde{X}_{2m-1}(t)| \geq |X_{2m-1}(t)| = 1$. (Moreover, it is easy to see that

$$\lim_{t\to 0}|\tilde{X}_{2m-1}(t)| = 1. \tag{3.7}$$

In fact, if $m = 1$ and if (ρ,θ) is the polar coordinates of M_2 at p_2 in which $\gamma(t)$ has the coordinates $(t,0)$ and the metric $g_{M_2} = d\rho^2 + G(\rho,\theta)d\theta^2$, then we have that $\frac{|\tilde{X}_{2m-1}(t)|}{|X_{2m-1}(t)|} = \frac{\sin t}{\sqrt{G(t,0)}}$. So far, the proof of (and comments on) Claim 2 is finished.)

Now, we consider $f_{[p_1\gamma(t)]}$ and $\mathbb{S}^{2m}_{[p_1\gamma(t)]}$. Let $\bar{\beta}_t(s)$ denote $f_{[p_1\gamma(t)]}(\beta_t(s))$, which is a minimal geodesic in $\mathbb{S}^{2m}_{[p_1\gamma(t)]}$ by (2.7). We can also define the corresponding

$\mathrm{d}f_{[p_1\gamma(t)]}$, which is an isometrical embedding (similar to Lemma 2.6). Hence, $\bar{\beta}_t'(0)$ is perpendicular to $\tilde{X}_1(t), \cdots, \tilde{X}_{2m-2}(t)$ and $\tilde{\gamma}'(t)$. On the other hand, note that $\tilde{X}_1(t), \cdots, \tilde{X}_{2m-2}(t)$, $\frac{\tilde{X}_{2m-1}(t)}{|\tilde{X}_{2m-1}(t)|}$ and $\tilde{\gamma}'(t)$ are parallel and orthogonal along $\tilde{\gamma}(t)$ (on $\mathbb{S}^{2m}_{[p_1p_2]} \subset \mathbb{S}^{2m+1}_{p_1}$). Then we can define an orientable angle function $\theta(t) \in (-\pi, \pi]$ between $\bar{\beta}_t'(0)$ and $\tilde{X}_{2m-1}(t)$. Note that

$$\theta(t) \neq 0, \pi \text{ for } t > 0 \tag{3.8}$$

(otherwise it has to hold that $\mathbb{S}^{2m}_{[p_1\gamma(t)]} = \mathbb{S}^{2m}_{[p_1p_2]}$, which contradicts (3.5)).
Claim 3: *We have that*

$$\cos\theta(t) = \frac{|X_{2m-1}(t)|}{|\tilde{X}_{2m-1}(t)|} = \frac{1}{|\tilde{X}_{2m-1}(t)|}. \tag{3.9}$$

As a corollary, $\theta(t)$ is a smooth function (which implies that $0 < \theta(t) < \frac{\pi}{2}$ (or $-\frac{\pi}{2} < \theta(t) < 0$) and $|\tilde{X}_{2m-1}(t)| > 1$ for $t > 0$ (see (3.8)). By Lemma 3.11, we first note that

$$\lim_{s\to 0} \left|\uparrow_{\bar{\beta}_t(s)}^{\tilde{\beta}_t(s)} \uparrow_{\bar{\beta}_t(s)}^{\tilde{\gamma}(t)}\right| = \frac{\pi}{2}.$$

Then it is easy to see that

$$\cos\theta(t) = \lim_{s\to 0} \frac{|\bar{\beta}_t(s)\tilde{\gamma}(t)|}{|\tilde{\beta}_t(s)\tilde{\gamma}(t)|} = \frac{|X_{2m-1}(t)|}{|\tilde{X}_{2m-1}(t)|} \quad \text{(i.e. (3.9) holds)}$$

(note that $f_{[p_1\gamma(t)]} : \beta_t(s)|_{s\in[0,\epsilon)} \to \bar{\beta}_t(s)|_{s\in[0,\epsilon)}$ is an isometry). Due to (3.9), in order to prove that $\theta(t)$ is a smooth function we need only to show that it is a continuous one. We first observe that $\theta(t) \to 0$ as $t \to 0$ because $\lim\limits_{t\to 0} |\tilde{X}_{2m-1}(t)| = 1$ (see (3.7)). Moreover, note that $\theta(t + \Delta t) = \theta(t) \pm \theta_{\tilde{\gamma}(t)}(\Delta t)$, where $\theta_{\tilde{\gamma}(t)}(\Delta t)$ denotes the angle between $\bar{\beta}_{t+\Delta t}'(0)$ and the vector (at $\tilde{\gamma}(t+\Delta t)$) that is parallel to $\bar{\beta}_t'(0)$ along $\tilde{\gamma}$. Similar to $\theta(t) \to 0$ as $t \to 0$, we have that $\theta_{\tilde{\gamma}(t)}(\Delta t) \to 0$ as $\Delta t \to 0$. That is, $\theta(t)$ is continuous with respect to t. (Now, Claim 3 is verified.)

Based on Claim 3, we have the following important observation.
Claim 4: *For any $q \in M_2$, $f_{p_1}(q)$ is a closed 1-dimensional smooth submanifold in $\mathbb{S}^{2m+1}_{p_1}$, i.e. $f_{p_1}(q)$ consists of finite smooth circles which do not intersect each other (and each of which does not intersect itself).* (Of course, when the whole proof of the lemma has been finished, we will know that $f_{p_1}(q)$ is just one or two great circles in $\mathbb{S}^{2m+1}_{p_1}$.)
According to Proposition 3.1, there are $[qr], [rr'], [qr'] \subset M_2$ with $|rq|, |rr'| < \frac{r_0}{2}$ such that the triangle formed by them does not bound a convex spherical surface. Without loss of generality, we can assume that $[rr']$ is just the $[p_2p_2']$. Then, due to (3.2), we have that $f_{[p_1p_2]}(q) \notin \mathbb{S}^{2m-1}_{[p_1p_2],[p_2p_2']}$. According to (3.5), we have that $f_{[p_1\gamma(t)]}(q) \notin \mathbb{S}^{2m-1}_{[p_1p_2],[p_2p_2']}$ either. Note that $\gamma(t)$ belongs to $B_{M_2}(q, r_0)$ for all t. It then follows from Claim 3 that

$$\alpha_q(t)|_{t\in[0,|p_2p_2'|]} \triangleq f_{[p_1\gamma(t)]}(q)|_{t\in[0,|p_2p_2'|]} \text{ is a smooth curve.} \tag{3.10}$$

Note that $\alpha_q(t_1) \neq \alpha_q(t_2)$ for all $t_1 \neq t_2$ by (3.5) (note that $f_{[p_1\gamma(t)]}(q) \notin \mathbb{S}^{2m-1}_{[p_1p_2],[p_2p_2']}$), so there is an interval $[a, b] \subseteq [0, |p_2p_2'|]$ such that the tangent vector

$$\alpha_q'(t) \neq 0 \text{ for all } t \in [a, b].$$

By Lemma 3.11, $\alpha'_q(t)$ is perpendicular to $\mathbb{S}^{2m}_{[p_1q]_t}$ for all $t\in[a,b]$, where $[p_1q]_t$ is the minimal geodesic between p_1 and q whose unit tangent vector at p_1 is $\alpha_q(t)$. Moreover, note that

$$B_{\mathbb{S}^{2m}_{[p_1q]_t}}(\alpha_q(t),r_0)\cap B_{\mathbb{S}^{2m}_{[p_1q]_{t'}}}(\alpha_q(t'),r_0)=\emptyset \text{ for all } t\neq t'\in[a,b]$$

(this is due to that $\alpha_q(t)\neq\alpha_q(t')$ and that $f_{[p_1\bar p_2]}|_{B_{M_2}(\bar p_2,r_0)}$ is injective for any $[p_1\bar p_2]$ with $\bar p_2\in M_2$ (see Remark 2.8)). Then it is easy to see that the r_0-tubler neighborhood U of $\alpha_q(t)|_{t\in[a,b]}$ satisfies

$$U=\bigcup_{t\in[a,b]}B_{\mathbb{S}^{2m}_{[p_1q]_t}}(\alpha_q(t),r_0) \text{ and } U\cap f_{p_1}(q)=\alpha_q(t)|_{t\in[a,b]}. \tag{3.11}$$

On the other hand, for any $\tilde q'\in f_{p_1}(q)$, there is a minimal geodesic $[p_1p_2]'$ between p_1 and p_2 such that $f_{[p_1p_2]'}(q)=\tilde q'$. We also consider $\mathbb{S}^{2m}_{[p_1p_2]'}$, the minimal geodesic $[p_1\gamma(t)]'$ between p_1 and $\gamma(t)$ whose unit tangent vector at p_1 is $f_{[p_1p_2]'}(\gamma(t))$, and the curve $\bar\alpha_q(t)|_{t\in[0,|p_2p_2'|]}\triangleq f_{[p_1\gamma(t)]'}(q)|_{t\in[0,|p_2p_2'|]}$ (with $\bar\alpha(0)=\tilde q'$). Note that

$$\bar\alpha_q(t)|_{t\in[0,|p_2p_2'|]} \text{ is identical to } \alpha_q(t)|_{t\in[0,|p_2p_2'|]} \text{ up to an isometry of } \mathbb{S}^{2m+1}_{p_1}. \tag{3.12}$$

This together with (3.11) implies that $f_{p_1}(q)$ is a closed 1-dimensional submanifold in $\mathbb{S}^{2m+1}_{p_1}$. (The proof of Claim 4 is done.)

Based on Claim 4, we can draw an almost immediate conclusion.

Claim 5: *$f_{p_1}:M_2\to\mathbb{S}^{2m+1}_{p_1}$ is surjective.* Let $\mathcal{S}$ be a component of $f_{p_1}(q)$ for the q in the proof of Claim 4, which is a smooth circle in $\mathbb{S}^{2m+1}_{p_1}$. For any $\tilde z\in\mathbb{S}^{2m+1}_{p_1}$, there is a $\tilde q\in\mathcal{S}$ such that $|\tilde z\tilde q|=\min\{|\tilde z\tilde q'||\tilde q'\in\mathcal{S}\}$. Note that $[\tilde q\tilde z]$ is perpendicular to $\mathcal{S}$ at $\tilde q$. On the other hand, there is a $[p_1q]$ such that $\tilde q=\uparrow_{p_1}^q$, and $\mathcal{S}$ is perpendicular to $\mathbb{S}^{2m}_{[p_1q]}$ at $\tilde q$ (by Lemma 3.11). It then follows that $\tilde z$ belongs to $\mathbb{S}^{2m}_{[p_1q]}\subset f_{p_1}(M_2)$ (see (2.10.1)).

We still consider the $\mathcal{S}$ in the proof of Claim 5. Let $s\in[0,\ell]$ be the arc-length parameter of $\mathcal{S}$, where ℓ is the perimeter of $\mathcal{S}$. It follows from the proof Claim 5 that

$$\mathbb{S}^{2m+1}_{p_1}=\bigcup_{s\in[0,\ell]}\mathbb{S}^{2m}_{[p_1q]_s},$$

where $[p_1q]_s$ is the minimal geodesic between p_1 and q whose unit tangent vector at p_1 is $\mathcal{S}(s)$. Note that there is a natural isometry (ref. Remark 2.8)

$$h_{s,s'}\triangleq\exp_{\mathcal{S}(s')}\circ\, \mathrm{d}f_{[p_1q]_{s'}}\circ \mathrm{d}f^{-1}_{[p_1q]_s}\circ\exp^{-1}_{\mathcal{S}(s)}:\mathbb{S}^{2m}_{[p_1q]_s}\to\mathbb{S}^{2m}_{[p_1q]_{s'}}. \tag{3.13}$$

By Lemma 2.4 (and (2.5)), we observe that $h_{s,s'}\to h_{s,s_0}$ if $s'\to s_0$, and thus $h_{0,s}(\tilde x)|_{s\in[0,\ell]}$ is continuous with respect to s for any $\tilde x\in\mathbb{S}^{2m}_{[p_1q]_0}$. Observe that

$$h_{0,s}(\tilde x)\neq h_{0,s'}(\tilde x) \text{ for any } 0\le s\neq s'<\ell. \tag{3.14}$$

(Otherwise, $h_{0,s}([\mathcal{S}(0)\tilde x])$ and $h_{0,s'}([\mathcal{S}(0)\tilde x])$ are two minimal geodesics starting from $h_{0,s}(\tilde x)$ in $\mathbb{S}^{2m}_{[p_1x]_s}$, where $x=f^{-1}_{p_1}(\tilde x)$ and $[p_1x]_s$ is the minimal geodesic between p_1 and

x whose unit tangent vector at p_1 is $h_{0,s}(\tilde{x})$. However, note that $f_{p_1}^{-1}(h_{0,s}([\mathcal{S}(0)\tilde{x}])) = f_{p_1}^{-1}(h_{0,s'}([\mathcal{S}(0)\tilde{x}])) = f_{p_1}^{-1}([\mathcal{S}(0)\tilde{x}])$. We know that this is impossible by Remark 2.8 when we consider $f_{[p_1x]_s}$.) It follows that

$$h_{0,s}(\tilde{x})|_{s\in[0,\ell]} \text{ is a component of } f_{p_1}(x). \tag{3.15}$$

On the other hand, note that there is a neighborhood V of q in $B_{M_2}(p_2, \frac{r_0}{2})$ such that $f_{[p_1p_2]}(V) \cap \mathbb{S}^{2m-1}_{[p_1p_2],[p_2p_2']} = \emptyset$ (because $f_{[p_1p_2]}(q) \notin \mathbb{S}^{2m-1}_{[p_1p_2],[p_2p_2']}$). Similar to $\alpha_q(t)$ and $\bar{\alpha}_q(t)$, we can define $\alpha_v(t)|_{t\in[0,|p_2p_2'|]} \triangleq f_{[p_1\gamma(t)]}(v)|_{t\in[0,|p_2p_2'|]}$ and $\bar{\alpha}_v(t)|_{t\in[0,|p_2p_2'|]} \triangleq f_{[p_1\gamma(t)]'}(v)|_{t\in[0,|p_2p_2'|]}$ for any $v \in V$. In fact, for any $v_1, v_2 \in V$, we have a strengthened version of (3.12) (note that the isometry of $\mathbb{S}^{2m+1}_{p_1}$ mentioned in (3.12) restricted to $\mathbb{S}^{2m}_{[p_1p_2]}$ is actually an isometry from $\mathbb{S}^{2m}_{[p_1p_2]}$ to $\mathbb{S}^{2m}_{[p_1p_2]'}$ like (3.13)):

$$|\alpha_{v_1}(t_1)\alpha_{v_2}(t_2)| = |\bar{\alpha}_{v_1}(t_1)\bar{\alpha}_{v_2}(t_2)| \text{ for all } t_1, t_2 \in [0, |p_2p_2'|]. \tag{3.16}$$

(This is a very important observation to the whole proof.)

Based on (3.16), we can conclude that: if $0 < \Delta s < \delta$ for a small δ, then the map

$$h_{0,s}(\tilde{x})|_{s\in[0,\Delta s]} \to h_{0,s}(\tilde{x})|_{s\in[\Delta s,2\Delta s]} \text{ defined by } h_{0,s}(\tilde{x}) \mapsto h_{s,s+\Delta s}(h_{0,s}(\tilde{x})) \text{ is an isometry} \tag{3.17}$$

for all $\tilde{x} \in \mathbb{S}^{2m}_{[p_1q]_0}$ (note that s is the arc-length parameter of $\mathcal{S}$). Let $\mathcal{S}^*$ denote the circle $h_{0,s}(\tilde{x})|_{s\in[0,\ell]}$ (see (3.15) and Claim 4), and let s^* be its arc-length parameter. Note that we can assume that $\mathcal{S}^*(0) = h_{0,0}(\tilde{x}) = \tilde{x}$, and $\mathcal{S}^*(\Delta s^*) = h_{0,\Delta s}(\tilde{x})$ for some $\Delta s^* > 0$. It then follows from (3.17) and (3.14) that

$$h_{0,s}(\tilde{x})|_{s\in[\Delta s,2\Delta s]} \text{ is the arc } \mathcal{S}^*(s^*)|_{s^*\in[\Delta s^*,2\Delta s^*]} \text{ with } h_{0,2\Delta s}(\tilde{x}) = \mathcal{S}^*(2\Delta s^*)$$

(NOT the arc $\mathcal{S}^*(s^*)|_{s^*\in[0,\Delta s^*]}$). Similarly, we have that $h_{0,k\Delta s}(\tilde{x}) = \mathcal{S}^*(k\Delta s^*)$ for any $k \in \mathbb{N}^+$. Due to the arbitrariness of Δs, this implies that

$$\frac{\Delta s^*}{\Delta s} = \frac{\ell^*}{\ell} \text{ and } h_{0,s}(\tilde{x}) = \mathcal{S}^*(s^*) \text{ with } \frac{s^*}{s} = \frac{\ell^*}{\ell} \text{ for any } s \in [0, \ell], \tag{3.18}$$

where ℓ^* is the perimeter of $\mathcal{S}^*$. Consequently, if $h_{0,s_0}(\tilde{x}) = \mathcal{S}^*(s_0^*)$, then

$$h_{s_0,s_0+s}(\mathcal{S}^*(s_0^*)) = \mathcal{S}^*(s_0^* + \frac{\ell^*}{\ell}s). \tag{3.19}$$

An important fact is that if $\mathcal{S}^*(s_0^*) \in \mathbb{S}^{2m}_{[p_1q]_0}$ for some $0 < s_0^* < \ell^*$, we also have that

$$h_{0,s}(\mathcal{S}^*(s_0^*)) = \mathcal{S}^*(s_0^* + \frac{\ell^*}{\ell}s). \tag{3.20}$$

If this is not true, then, by (3.18), it has to hold that $h_{0,s}(\mathcal{S}^*(s_0^*)) = \mathcal{S}^*(s_0^* - \frac{\ell^*}{\ell}s)$, which implies that $h_{0,\frac{s_0}{2}}(\tilde{x}) = h_{0,\frac{s_0}{2}}(\mathcal{S}^*(s_0^*)) = \mathcal{S}^*(\frac{s_0^*}{2})$. Since $h_{0,\frac{s_0}{2}}$ is an isometry, we have that $\tilde{x} = \mathcal{S}^*(s_0^*) = h_{0,s_0}(\tilde{x})$, which contradicts (3.14). On the other hand, by Claims 5 and 4 we know that

$$\mathbb{S}^{2m+1}_{p_1} = \bigcup_{p_2\in M_2} f_{p_1}(p_2),$$

where each $f_{p_1}(p_2)$ consists of finite components similar to $\mathcal{S}^*$. Moreover, by the proof of Claim 5, there exist a $\tilde{z}_0 \in \mathbb{S}^{2m}_{[p_1q]_0}$ and $s_{\tilde{z}_0}$ such that $h_{0,s_{\tilde{z}_0}}(\tilde{z}_0) = \tilde{z}$ for any $\tilde{z} \in \mathbb{S}^{2m+1}_{p_1}$. Therefore, based on $\mathcal{S}$ and due to (3.19-20), we can define an S^1-action on $\mathbb{S}^{2m+1}_{p_1}$

$$h : \mathcal{S} \times \mathbb{S}^{2m+1}_{p_1} \to \mathbb{S}^{2m+1}_{p_1} \text{ defined by } h(s, \tilde{z}) = h_{0,s_{\tilde{z}_0}+s}(\tilde{z}_0), \tag{3.21}$$

where $\tilde{z}_0 \in \mathbb{S}^{2m}_{[p_1q]_0}$ and $h_{0,s_{\tilde{z}_0}}(\tilde{z}_0) = \tilde{z}$.

Claim 6: *Through h, $\mathcal{S}$ acts on $\mathbb{S}^{2m+1}_{p_1}$ freely and isometrically.* By (3.14), $\mathcal{S}$ acts on $\mathbb{S}^{2m+1}_{p_1}$ freely. It then suffices to show that each $h(s, \cdot)$ is a local isometry (note that $h(s, \cdot)$ is a 1-1 map). For any $\tilde{z} \in \mathbb{S}^{2m+1}_{p_1}$ and $z \triangleq f^{-1}_{p_1}(\tilde{z})$, we let $\mathcal{S}_{\tilde{z}}$ be the component of $f_{p_1}(z)$ containing $\tilde{z}$. We first assume that $\tilde{z}$ is sufficiently close to some $\mathcal{S}(s_0)$, and will prove that

$$|h(s, \mathcal{S}(s_0))h(s, \tilde{z})| = |\mathcal{S}(s_0)\tilde{z}|. \tag{3.22}$$

Note that $h(s, \mathcal{S}(s_0)) = \mathcal{S}(s_0 + s)$ and $h(s, \tilde{z}) = h_{0,s_{\tilde{z}_0}+s}(\tilde{z}_0)$, and that $\tilde{z}$ and $h(s, \tilde{z})$ are the unique points in $\mathcal{S}_{\tilde{z}}$ such that

$$|\mathcal{S}(s_{\tilde{z}_0})\tilde{z}| = |\mathcal{S}(s_{\tilde{z}_0} + s)h(s, \tilde{z})| = |\mathcal{S}(0)\tilde{z}_0| \ (\leq |\mathcal{S}(s_0)\tilde{z}|)$$

(see Remark 2.8). Of course, this implies that $|\mathcal{S}(s_{\tilde{z}_0})\mathcal{S}(s_0)|$ is sufficiently small, so is $|s_{\tilde{z}_0} - s_0|$. It then is not hard to see that (3.22) follows from (3.16). Now we let $\tilde{z}$ be an arbitrary point in $\mathbb{S}^{2m+1}_{p_1}$, and we need only to show that

$$|h(s, \tilde{z}')h(s, \tilde{z})| = |\tilde{z}'\tilde{z}| \tag{3.23}$$

for any $\tilde{z}'$ in $\mathbb{S}^{2m+1}_{p_1}$ sufficiently close to $\tilde{z}$. Let s^* (resp. s^{**}) be the arc-length parameter of $\mathcal{S}_z$ (resp. $\mathcal{S}_{z'}$), which is increasing with respect to s, such that $h(s, \tilde{z}) = \mathcal{S}_{\tilde{z}}(s^*)$ with $\mathcal{S}_{\tilde{z}}(0) = \tilde{z}$ (resp. $h(s, \tilde{z}') = \mathcal{S}_{\tilde{z}'}(s^{**})$ with $\mathcal{S}_{\tilde{z}'}(0) = \tilde{z}'$). If we replace $\mathcal{S}$ with $\mathcal{S}_{\tilde{z}}$, we can similarly define $\bar{h}(s^*, \cdot)$ such that $\bar{h}(s^*, \tilde{z}) = \mathcal{S}_{\tilde{z}}(s^*)$. Similarly, we have that $\bar{h}(s^*, \tilde{z}') = \mathcal{S}_{\tilde{z}'}(s^{**\prime})$ with $\mathcal{S}_{\tilde{z}'}(s^{**\prime})|_{s^{**\prime}=0} = \tilde{z}'$, where $s^{**\prime}$ is another arc-length parameter of $\mathcal{S}_{\tilde{z}'}$; and by (3.22) we have that

$$|\bar{h}(s^*, \tilde{z})\bar{h}(s^*, \tilde{z}') = |\tilde{z}\tilde{z}'|. \tag{3.24}$$

Note that $s^{**\prime} = s^{**}$ or $s^{**\prime} = -s^{**}$, and it will suffice to show that the latter case does not occur. In fact, if $s^{**\prime} = -s^{**}$, then (3.24) implies that $|h(s, \tilde{z}_1)h(s, \tilde{z}')|$ will change as s changes, where $\tilde{z}_1$ is a point in $\mathcal{S}_{\tilde{z}}$ such that $\tilde{z}_1$ and $\tilde{z}'$ lie in some $\mathbb{S}^{2m}_{[p_1q]_{s_1}}$. This is impossible because $|h(s, \tilde{z}_1)h(s, \tilde{z}')| = |h_{s_1,s_1+s}(\tilde{z}_1)h_{s_1,s_1+s}(\tilde{z}')|$ and h_{s_1,s_1+s} is an isometry. Note that the proof of Claim 6 is completed now.

By Claim 6, we know that $\mathbb{S}^{2m+1}_{p_1}/\mathcal{S} = \mathbb{CP}^m$. And it is easy to see that

$$\mathbb{S}^{2m+1}_{p_1}/\mathcal{S}(= \mathbb{CP}^m) \to M_2 \text{ defined by } \{f_{p_1}(p_2)\} \mapsto p_2 \tag{3.25}$$

is a locally isometrical map (see (2.2) and Lemma 2.3), i.e. it is a Riemannian covering map. Therefore, according to Synge's theorem ([CE]), M_2 is isometric to $\mathbb{CP}^m$ or

$\mathbb{CP}^m/\mathbb{Z}_2$ with the canonical metric, and that $M_2 \overset{\text{iso}}{\cong} \mathbb{CP}^m/\mathbb{Z}_2$ occurs only when m is odd (see Lemma A.1 in Appendix). (The long proof of Lemma 3.10 is completed now.) $\square$

Based on Lemma 3.10 and its proof, we give the following two important facts. In the following, we assume that $n_i = 2m_i$ for $i = 1$ and 2 (see Lemma 3.8).

Lemma 3.12. *For any $p \in M$, there exists a $[p_1p_2]$ with $p_i \in M_i$ such that $p \in [p_1p_2]$.*

Proof. We select $p_1 \in M_1$ such that $|pp_1| = \min\{|pp_1'||p_1' \in M_1\}$. Note that for any $[p_1p]$, we have that $\uparrow_{p_1}^{p} \in (\Sigma_{p_1}M_1)^{=\frac{\pi}{2}}$ $(= \mathbb{S}_{p_1}^{2m_2+1})$. By Claim 5 in the proof of Lemma 3.10, there is a $[p_1p_2]$ with $p_2 \in M_2$ such that

$$\uparrow_{p_1}^{p} = \uparrow_{p_1}^{p_2}.$$

Claim: $|pp_1| \le \frac{\pi}{2}$, *and thus $p \in [p_1p_2]$*. If $|pp_1| > \frac{\pi}{2}$, then $p_2 \in [p_1p]$, and so $|pp_2| \le \frac{\pi}{2}$ (because $|p_1p| \le \pi$). Note that $|pp_2| = \min\{|pp_2'||p_2' \in M_2\}$ because $|p_1p_2'| = \frac{\pi}{2}$ for all $p_2' \in M_2$. Similarly, if

$$f_{p_2}(M_1) = (\Sigma_{p_2}M_2)^{=\frac{\pi}{2}}, \tag{3.26}$$

then we can find a $[p_2\bar{p}_1]$ with $\bar{p}_1 \in M_1$ such that $p \in [p_2\bar{p}_1]$, which contradicts the choice of p_1. Hence, we need only to prove (3.26). If $n_1 = 2m_1 \ge 2$, then (3.26) automatically holds (similar to Claim 5 in the proof of Lemma 3.10). If $m_1 = 0$ (i.e. $M_1 = \{p_1\}$), there is a natural map

$$\tau : f_{p_1}(p_2) \to (\Sigma_{p_2}M_2)^{=\frac{\pi}{2}} \text{ defined by } \uparrow_{p_1}^{p_2} \mapsto \uparrow_{p_2}^{p_1},$$

where $\uparrow_{p_1}^{p_2}$ and $\uparrow_{p_2}^{p_1}$ are the directions of any given $[p_1p_2]$. Obviously, τ is injective and continuous. On the other hand, note that $(\Sigma_{p_2}M_2)^{=\frac{\pi}{2}}$ is a circle (in this case M_2 is of codimension 2), and each component of $f_{p_1}(p_2)$ is a circle (see Claim 4 in the proof of Lemma 3.10). Hence, τ restricted to each component of $f_{p_1}(p_2)$ is surjective, and thus (3.26) follows (and $f_{p_1}(p_2)$ contains only one component). $\square$

Lemma 3.13. *If M_2 is isometric to $\mathbb{CP}^{m_2}$ (here m_2 may be 0), then*
(3.13.1) *M_1 is isometric to $\mathbb{CP}^{m_1}$.*
(3.13.2) *For any $p \in M$, $\{p\}^{=\frac{\pi}{2}}$ is totally geodesic and of codimension 2 in M, and so it is isometric to $\mathbb{CP}^{\frac{n}{2}-1}$.*

Proof. (3.13.1) We need only to consider the case $m_1 > 0$. We consider the following natural map

$$\bar{\tau} : f_{p_2}(p_1) \to f_{p_1}(p_2) \text{ defined by } \uparrow_{p_2}^{p_1} \mapsto \uparrow_{p_1}^{p_2}$$

(similar to the τ in the proof of Lemma 3.12). Since M_2 is isometric to $\mathbb{CP}^{m_2}$, from the end of the proof of Lemma 3.10 (resp. Lemma 3.12), we can conclude that $f_{p_1}(p_2)$ is just a circle when $m_2 > 0$ (resp. $m_2 = 0$). Then by the arguments in the end of the

proof of Lemma 3.12, $f_{p_2}(p_1)$ contains only one component (so it is a circle too). This together with Lemma 3.10 (and the end of its proof) implies that $M_1 \overset{\text{iso}}{\cong} \mathbb{CP}^{m_1}$.

(3.13.2) We first supply the proof for any given point $p \in M_i$, say M_2. Since $M_2 \overset{\text{iso}}{\cong} \mathbb{CP}^{m_2}$, $N_2 \triangleq \{p\}^{=\frac{\pi}{2}} \cap M_2$ is isometric to $\mathbb{CP}^{m_2-1}$ and is totally geodesic in M_2 (note that we need only to consider the case "$m_2 > 0$"). Then, by Lemma 3.12 and 2.3, it is not hard to see that

$$\{p\}^{=\frac{\pi}{2}} = \{q \in M | q \text{ belongs to some } [p_1p_2] \text{ with } p_1 \in M_1 \text{ and } p_2 \in N_2\}$$

and that

$$\{p\}^{=\frac{\pi}{2}} = \{p\}^{\geq\frac{\pi}{2}}. \tag{3.27}$$

Moreover, we have that $\lambda_{pq} = +\infty$ for any $q \in \{p\}^{=\frac{\pi}{2}}$. Hence, by (3.13.1) we can get that $\{p\}^{=\frac{\pi}{2}}$ is isometric to $\mathbb{CP}^{\frac{n}{2}-1}$ once we have proved that it is totally geodesic (note that it is closed in M) and of codimension 2 in M.

Note that $\{p\}^{\geq\frac{\pi}{2}}$ is convex in M by (i) of Theorem 1.1. Then (3.27) and "$\lambda_{pq} = +\infty\ \forall\ q \in \{p\}^{=\frac{\pi}{2}}$" imply that $\dim(\{p\}^{=\frac{\pi}{2}}) \leq n-2$ (ref. [RW][5]). On the other hand, note that both M_1 and N_2 are totally geodesic in $\{p\}^{=\frac{\pi}{2}}$, and that $|p_1p_2| = \frac{\pi}{2}$ and $\lambda_{p_1p_2} = +\infty$ for all $p_1 \in M_1$ and $p_2 \in N_2$. Similarly, it is implied that $\dim(\{p\}^{=\frac{\pi}{2}}) \geq n-2$. It then follows that $\dim(\{p\}^{=\frac{\pi}{2}}) = n-2$.

Next we will prove that $\{p\}^{=\frac{\pi}{2}}$ has empty boundary, which implies that $\{p\}^{=\frac{\pi}{2}}$ is totally geodesic (because it is convex) in M. Since any $q \in \{p\}^{=\frac{\pi}{2}}$ lies in a $[p_1p_2]$ with $p_1 \in M_1$ and $p_2 \in N_2$, it suffices to show that both M_1 and N_2 consist of interior points[6] of $\{p\}^{=\frac{\pi}{2}}$. Let p_2 be an arbitrary point in N_2. By (3.26), $(\Sigma_{p_2}M_2)^{=\frac{\pi}{2}}$ belongs to $\Sigma_{p_2}\{p\}^{=\frac{\pi}{2}}$. Furthermore, by Lemma 2.3 it is easy to see that

$$\Sigma_{p_2}\{p\}^{=\frac{\pi}{2}} = (\Sigma_{p_2}M_2)^{=\frac{\pi}{2}} * \Sigma_{p_2}N_2 = \mathbb{S}^{n-3}$$

(note that N_2 is totally geodesic in $\{p\}^{=\frac{\pi}{2}}$). It follows that p_2 ($\in N_2$) is an interior point of $\{p\}^{=\frac{\pi}{2}}$. Let p_1 be an arbitrary point in M_1. From (3.25), it is easy to see that $f_{p_1}(N_2)$ is an $\mathbb{S}^{2m_2-1}$ in $\mathbb{S}^{2m_2+1}_{p_1}$. Then similarly, we can get that $\Sigma_{p_1}\{p\}^{=\frac{\pi}{2}} = f_{p_1}(N_2) * \Sigma_{p_1}M_1 = \mathbb{S}^{n-3}$, i.e. p_1 is also an interior point of $\{p\}^{=\frac{\pi}{2}}$.

So far we have given the proof for any point $p \in M_2$. Now let p be an arbitrary point in M. By Lemma 3.12, there is a $[p_1p_2]$ with $p_i \in M_i$ such that $p \in [p_1p_2]$. Since $M_i \overset{\text{iso}}{\cong} \mathbb{CP}^{m_i}$ with m_1 or $m_2 > 0$, say $m_2 > 0$, we can select $\bar{p}_2 \in M_2$ such that $|\bar{p}_2p_2| = \frac{\pi}{2}$. We have proved that $\{\bar{p}_2\}^{=\frac{\pi}{2}}$ is isometric to $\mathbb{CP}^{\frac{n}{2}-1}$, and it is easy to see that $p \in \{\bar{p}_2\}^{=\frac{\pi}{2}}$ by Lemma 2.3. Hence, (3.13.2) follows if we replace M_1 and M_2 with $\bar{M}_1 \triangleq \{\bar{p}_2\}$ and $\bar{M}_2 \triangleq \{\bar{p}_2\}^{=\frac{\pi}{2}}$ respectively. $\square$

[5]In [RW], it has been proved that: *Let A_1 and A_2 be two convex subsets in an n-dimensional Alexandrov space with curvature ≥ 1. If $|a_1a_2| = \frac{\pi}{2}$ for any $a_i \in A_i$, then $\dim(A_1) + \dim(A_2) \leq n-1$; and if equality holds, then $\lambda_{a_1a_2} < +\infty$ for all $a_i \in A_i^\circ$ (where X° denotes the interior part of X).*

[6]We know that, in an Alexandrov space A with curvature bounded below, any minimal geodesic between two interior points belongs to A° (ref. [BGP]).

To the whole proof of the Main Theorem, the most difficult parts are to prove that M_i ($i = 1, 2$) are both isometric to $\mathbb{CP}^{m_i}$ or $\mathbb{CP}^{m_i}/\mathbb{Z}_2$, and that any $p \in M$ lies in a $[p_1p_2]$ with $p_i \in M_i$. We would like to point out that once these are established, we can find an argument in [GG1] to prove that M isometric to $\mathbb{CP}^{\frac{n}{2}}$ or $\mathbb{CP}^{\frac{n}{2}}/\mathbb{Z}_2$, i.e. the following lemma holds. For the convenience of readers and the completeness of the present paper, we will supply a detailed proof for it.

Lemma 3.14. *If $\lambda_{p_1p_2} = +\infty$ for all $p_i \in M_i$, then $M_i \overset{\text{iso}}{\cong} \mathbb{CP}^{m_i}$ and $M \overset{\text{iso}}{\cong} \mathbb{CP}^{\frac{n}{2}}$, or $M_i \overset{\text{iso}}{\cong} \mathbb{CP}^{m_i}/\mathbb{Z}_2$ and $M \overset{\text{iso}}{\cong} \mathbb{CP}^{\frac{n}{2}}/\mathbb{Z}_2$ (only when m_i and $\frac{n}{2}$ are odd) with canonical metrics.*

Proof. By Lemma 3.10 and 3.13, M_i is isometric to $\mathbb{CP}^{m_i}$ or $\mathbb{CP}^{m_i}/\mathbb{Z}_2$ ($i = 1, 2$) with the canonical metric, and that $M_i \overset{\text{iso}}{\cong} \mathbb{CP}^{m_i}/\mathbb{Z}_2$ occurs only when m_1 and m_2 are odd. Then we can divide the proof into the following two cases.

Case 1. $M_i \overset{\text{iso}}{\cong} \mathbb{CP}^{m_i}$. In this case, we will prove that M is isometric to $\mathbb{CP}^{\frac{n}{2}}$.

According to (3.13.2), we can assume that $n_1 = 0$ (i.e. $M_1 = \{p_1\}$) and $n_2 = n - 2$, and thus $M_2 \overset{\text{iso}}{\cong} \mathbb{CP}^{\frac{n}{2}-1}$. Let $\nu : M_2 \hookrightarrow \mathbb{CP}^{\frac{n}{2}}$ be an isometrical embedding whose image is denoted by $\hat{M}_2$, and let $\hat{p}_1$ be the point in $\mathbb{CP}^{\frac{n}{2}}$ such that $d(\hat{p}_1, \hat{M}_2) = \frac{\pi}{2}$ (note that $|\hat{p}_1\hat{p}_2| = \frac{\pi}{2}$ for any $\hat{p}_2 \in \hat{M}_2$). Due to Claim 5 and (3.25) in the proof of Lemma 3.10, $\Sigma_{p_1}M$ (resp. $\Sigma_{\hat{p}_1}\mathbb{CP}^{\frac{n}{2}}$) admits an isometrical and free S^1-action such that each S^1-orbit is some $\Uparrow_{p_1}^{p_2}$ (resp. $\Uparrow_{\hat{p}_1}^{\hat{p}_2}$) and $(\Sigma_{p_1}M)/S^1 = M_2$ (resp. $(\Sigma_{\hat{p}_1}\mathbb{CP}^{\frac{n}{2}})/S^1 = \hat{M}_2$). Hence, there is a natural isometrical map

$$\mathbf{i}_* : \Sigma_{p_1}M \to \Sigma_{\hat{p}_1}\mathbb{CP}^{\frac{n}{2}}\ (= \mathbb{S}^{n-1})$$

such that $\mathbf{i}_*(\Uparrow_{p_1}^{p_2}) = \Uparrow_{\hat{p}_1}^{\hat{p}_2}$ with $\hat{p}_2 = \nu(p_2)$ for any $p_2 \in M_2$. Furthermore, due to Lemma 3.12, $\mathbf{i}_*$ induces a natural 1-1 map

$$\mathbf{i} : M \to \mathbb{CP}^{\frac{n}{2}}$$

with $\mathbf{i}(p_1) = \hat{p}_1$, $\mathbf{i}|_{M_2} = \nu$ and $\mathbf{i}([p_1p_2]) = [\hat{p}_1\hat{p}_2]$ for any $[p_1p_2]$ with $p_2 \in M_2$ such that $\uparrow_{\hat{p}_1}^{\hat{p}_2} = \mathbf{i}_*(\uparrow_{p_1}^{p_2})$ and $\mathbf{i}|_{[p_1p_2]}$ is an isometry. Let x and y be any two points in M. We need to show that

$$|\mathbf{i}(x)\mathbf{i}(y)| = |xy|. \tag{3.28}$$

By Lemma 3.12, we can select $[p_1p_x]$ and $[p_1p_y]$ with $p_x, p_y \in M_2$ such that $x \in [p_1p_x]$ and $y \in [p_1p_y]$. In the following, $\hat{p}$ always denotes $\mathbf{i}(p)$ for any $p \in M$. Note that $\hat{x} \in [\hat{p}_1\hat{p}_x]$ and $\hat{y} \in [\hat{p}_1\hat{p}_y]$ with $|\hat{x}\hat{p}_1| = |xp_1|$ and $|\hat{y}\hat{p}_1| = |yp_1|$. Since $M_2 \overset{\text{iso}}{\cong} \mathbb{CP}^{\frac{n}{2}-1}$, there is a $[p_xp_2] \subset M_2$ with $|p_xp_2| = \frac{\pi}{2}$ such that $p_y \in [p_xp_2]$. By (iii) of Theorem 1.1, there are $[p_1p_2]$ and another minimal geodesic $[p_1p_x]'$ between p_1 and p_x such that the triangle formed by $[p_1p_2]$, $[p_1p_x]'$ and $[p_xp_2]$ bounds a convex spherical surface which contains $[p_1p_y]$ (ref. [GM]). In this surface, there is a $[p_2y']$ with $y' \in [p_1p_x]'$ such that

$y \in [p_2y']$. And an important point is that, based on Lemma 2.4, it is not hard to see that $\hat{y}$ belongs to $[\hat{p}_2\hat{y'}]$ with $|\hat{y}\hat{y'}| = |yy'|$ (note that $\hat{y'} \in \mathbf{i}([p_1p_x]'))$.

Note that $x, y' \in \{p_2\}^{=\frac{\pi}{2}}$, so $[p_2y']$ is perpendicular to $\{p_2\}^{=\frac{\pi}{2}}$ at y' (note that $\{p_2\}^{=\frac{\pi}{2}}$ is totally geodesic in M by (3.13.2)). Then by Lemma 2.3, it is easy to see that

$$\cos|xy| = \cos|yy'| \cos|xy'|.$$

On the other hand, similarly, it is not hard to see that

$$\cos|\hat{x}\hat{y}| = \cos|\hat{y}\hat{y'}| \cos|\hat{x}\hat{y'}|$$

(with $|\hat{y}\hat{y'}| = |yy'|$). Hence, in order to see (3.28), it suffices to show that

$$|\hat{x}\hat{y'}| = |xy'|. \tag{3.29}$$

By the definition of $\mathbf{i}$, it is easy to see that $\mathbf{i}(\{p_2\}^{=\frac{\pi}{2}}) = \{\hat{p}_2\}^{=\frac{\pi}{2}}$. Moreover, by (3.13.2) we know that $\{p_2\}^{=\frac{\pi}{2}}$ is isometric to $\mathbb{CP}^{\frac{n}{2}-1}$, which together with (3.25) implies that $\mathbf{i}|_{\{p_2\}^{=\frac{\pi}{2}}}$ is an isometry. Hence, (3.29) follows (because $x, y' \in \{p_2\}^{=\frac{\pi}{2}}$).

Case 2. $M_i \overset{\text{iso}}{\cong} \mathbb{CP}^{m_i}/\mathbb{Z}_2$. In this case, $\frac{n}{2}$ is odd because both m_1 and m_2 are odd, and we will prove that M is isometric to $\mathbb{CP}^{\frac{n}{2}}/\mathbb{Z}_2$.

Note that M_i ($i = 1, 2$) can be embedded isometrically into $\mathbb{CP}^{\frac{n}{2}}/\mathbb{Z}_2$ with $|\hat{p}_1\hat{p}_2| = \frac{\pi}{2}$ for any $\hat{p}_i \in \hat{M}_i$ (see (A.1) in Appendix), where $\hat{M}_i$ (resp. $\hat{p}_i$) denotes the embedding image of M_i (resp. any given point $p_i \in M_i$). Similar to the $\mathbf{i}_*$ in Case 1, for a fixed point $p_{2,0} \in M_2$, there is an isometrical map

$$j_* : (\Sigma_{p_{2,0}} M_2)^{=\frac{\pi}{2}} \ (= \mathbb{S}^{2m_1+1}_{p_{2,0}}) \to (\Sigma_{\hat{p}_{2,0}} \hat{M}_2)^{=\frac{\pi}{2}} \ (= \mathbb{S}^{2m_1+1}_{\hat{p}_{2,0}})$$

with $j_*(\Uparrow_{p_{2,0}}^{p_1}) = \Uparrow_{\hat{p}_{2,0}}^{\hat{p}_1}$ for any $p_1 \in M_1$. Note that $j_*|_{\Uparrow_{p_{2,0}}^{p_1}}$ induces a natural homeomorphism $\bar{j}_{*,p_1} : \Uparrow_{p_1}^{p_{2,0}} \to \Uparrow_{\hat{p}_1}^{\hat{p}_{2,0}}$ which maps the unit tangent vector at p_1 of a $[p_1p_{2,0}]$ to that at $\hat{p}_1$ of the $[\hat{p}_1\hat{p}_{2,0}]$ with $\uparrow_{\hat{p}_{2,0}}^{\hat{p}_1} = j_*(\uparrow_{p_{2,0}}^{p_1})$. It is not hard to see that, for any $p_1 \in M_1$, there is a unique homeomorphism (ref. the $\mathbf{i}_*$ in Case 1)

$$i_{p_1*} : (\Sigma_{p_1} M_1)^{=\frac{\pi}{2}} \ (= \mathbb{S}^{2m_2+1}_{p_1}) \to (\Sigma_{\hat{p}_1} \hat{M}_1)^{=\frac{\pi}{2}} \ (= \mathbb{S}^{2m_2+1}_{\hat{p}_1})$$

such that $i_{p_1*}|_{\Uparrow_{p_1}^{p_{2,0}}} = \bar{j}_{*,p_1}$, $i_{p_1*}(\Uparrow_{p_1}^{p_2}) = \Uparrow_{\hat{p}_1}^{\hat{p}_2}$ for any $p_2 \in M_2$, and $|i_{p_1*}(\uparrow_{p_1}^{p_2}) i_{p_1*}(\uparrow_{p_1}^{p_{2,0}})| = |\uparrow_{p_1}^{p_2}\uparrow_{p_1}^{p_{2,0}}|$ if $|\uparrow_{p_1}^{p_2}\uparrow_{p_1}^{p_{2,0}}| = |p_2p_{2,0}|$. Then due to Lemma 3.12, there is a natural 1-1 map (similar to the $\mathbf{i}$ in Case 1)

$$\iota : M \to \mathbb{CP}^{\frac{n}{2}}/\mathbb{Z}_2$$

such that $\iota([p_1p_2]) = [\hat{p}_1\hat{p}_2]$ with $\uparrow_{\hat{p}_1}^{\hat{p}_2} = i_{p_1*}(\uparrow_{p_1}^{p_2})$ for any $[p_1p_2]$ with $p_i \in M_i$.

Claim: *ι is a continuous map (so it is a homeomorphism).* Note that $\iota|_{M_i}$ is an isometrical embedding for $i = 1$ and 2, and that $\iota(M_i) = \hat{M}_i$. Moreover, ι restricted to any convex spherical surface bounded by some $[p_1p_2]$, $[p_1p_{2,0}]$ and $[p_2p_{2,0}]$ (here $p_i \in M_i$) is an isometrical embedding (this is due to Lemma 2.4). It is not hard to see that these together with (2.5) imply that ι is a continuous map.

From the above claim, we know that $\pi_1(M) \cong \mathbb{Z}_2$. Let $\pi : \tilde{M} \to M$ be the Riemannian covering map. It suffices to show that $\tilde{M}$ is isometric to $\mathbb{CP}^{\frac{n}{2}}$. Since ι is a homeomorphism and $\iota(M_i) = \hat{M}_i$, we have that $\pi^{-1}(M_i)$ is connected (note that $\hat{M}_i$ $(= \mathbb{CP}^{m_i}/\mathbb{Z}_2) \hookrightarrow \mathbb{CP}^{\frac{n}{2}}/\mathbb{Z}_2$ induces an isomorphism from $\pi_1(\hat{M}_i)$ to $\pi_1(\mathbb{CP}^{\frac{n}{2}}/\mathbb{Z}_2)$), and thus

$$\pi^{-1}(M_i) \overset{\text{iso}}{\cong} \mathbb{CP}^{m_i}$$

because $M_i \overset{\text{iso}}{\cong} \mathbb{CP}^{m_i}/\mathbb{Z}_2$ and M_i is totally geodesic in M (which implies that $\pi^{-1}(M_i)$ is totally geodesic in $\tilde{M}$). Moreover, note that $|\tilde{p}_1\tilde{p}_2| \geq \frac{\pi}{2}$ for any $\tilde{p}_i \in \pi^{-1}(M_i)$. Hence, $\tilde{M}$ satisfies the conditions of the Main Theorem, so it follows from Case 1 that $\tilde{M}$ is isometric to $\mathbb{CP}^{\frac{n}{2}}$. □

Appendix

A.1. On $\mathbb{CP}^m/\mathbb{Z}_2$ (m is odd) with the canonical metric

How to get $\mathbb{CP}^m/\mathbb{Z}_2$? We know that

$$\mathbb{S}^{2m+1} = \{(z_1, \cdots, z_{m+1}) | z_i \in \mathbb{C},\ |z_1|^2 + \cdots + |z_{m+1}|^2 = 1\},$$

and S^1 can act on $\mathbb{S}^{2m+1}$ freely and isometrically (see (0.3)) through

$$S^1 \times \mathbb{S}^{2m+1} \to \mathbb{S}^{2m+1} \text{ defined by } (e^{i\theta}, (z_1, \cdots, z_{m+1})) \mapsto (e^{i\theta} z_1, \cdots, e^{i\theta} z_{m+1}).$$

And if m is odd, then there is an isometry of order 4 on $\mathbb{S}^{2m+1}$:

$$\varsigma : \mathbb{S}^{2m+1} \to \mathbb{S}^{2m+1} \text{ defined by } (z_1, z_2, \cdots, z_{2j-1}, z_{2j}, \cdots) \mapsto (-\bar{z}_2, \bar{z}_1, \cdots, -\bar{z}_{2j}, \bar{z}_{2j-1}, \cdots).$$

Note that ς induces a 2-order isometry $\hat{\varsigma}$ without fixed points on $\mathbb{S}^{2m+1}/S^1$. $(\mathbb{S}^{2m+1}/S^1)/\langle\varsigma\rangle$ endowed with the induced metric from the unit sphere $\mathbb{S}^{2m+1}$ is just the $\mathbb{CP}^m/\mathbb{Z}_2$ with the canonical metric. Moreover, for any odd $m_i > 0$ with $m_1 + m_2 = m - 1$, $\mathbb{CP}^{m_i}/\mathbb{Z}_2$ can be embedded isometrically into $\mathbb{CP}^m/\mathbb{Z}_2$ with

$$|q_1q_2| = \frac{\pi}{2} \text{ for any } q_i \in \mathbb{CP}^{m_i}/\mathbb{Z}_2. \tag{A.1}$$

In other words, $\mathbb{CP}^m/\mathbb{Z}_2$ is indeed an example of the Main Theorem.

As for even m, we have the following property.

Lemma A.1. *$\mathbb{Z}_2$ can* NOT *act on $\mathbb{CP}^m$ freely by isometries when m is even.*

Proof. We will give the proof by the induction on m. Obviously, when $m = 0$, this is true (because $\mathbb{CP}^0$ contains only one point). Now we assume that $m > 0$, and that $\mathbb{Z}_2 \triangleq \langle\sigma\rangle$ acts on $\mathbb{CP}^m$ by isometries. Let p be an arbitrary point in $\mathbb{CP}^m$. Note that $|p\sigma(p)| \leq \frac{\pi}{2}$, and that $\sigma([p\sigma(p)])$ is also a minimal geodesic between p and $\sigma(p)$ for any $[p\sigma(p)]$. It follows that σ fixes the middle point of $[p\sigma(p)]$ if $|p\sigma(p)| < \frac{\pi}{2}$ (because there is a unique minimal geodesic between p and $\sigma(p)$ when $|p\sigma(p)| < \frac{\pi}{2}$). If $|p\sigma(p)| = \frac{\pi}{2}$,

then σ preserves the set $L \triangleq \{x \in \mathbb{CP}^m | x \text{ belongs to some } [p\sigma(p)]\}$ which is isometric to $\mathbb{CP}^1$. Note that $L^{=\frac{\pi}{2}}$ is isometric to $\mathbb{CP}^{m-2}$ and totally geodesic (in $\mathbb{CP}^m$). Since σ preserves L, it has to preserves $L^{=\frac{\pi}{2}}$, and thus $\sigma|_{L^{=\frac{\pi}{2}}}$ is an isometry. By the inductive assumption, σ has fixed points on $L^{=\frac{\pi}{2}}$ (so on $\mathbb{CP}^m$). $\square$

In fact, $\mathbb{Z}_2$ cannot act on $\mathbb{CP}^m$ freely in the sense of topology when m is even ([Sa]).

A.2. Proof of (3.4.1) in Lemma 3.4 ([RW])

Since $\lambda_{p_1p_2} \equiv h$ for all $p_i \in M_i$, due to Lemma 2.3 it follows that, for the given $p_1 \in M_1$ and $p_2 \in M_2$, there are $\varepsilon > 0$ and a neighborhood V of p_2 in M_2 such that

$$\min_{1\leq j\neq j'\leq h}\{|(\uparrow_{p_1}^{p_2'})_j(\uparrow_{p_1}^{p_2'})_{j'}||(\uparrow_{p_1}^{p_2'})_j,(\uparrow_{p_1}^{p_2'})_{j'}\in\Uparrow_{p_1}^{p_2'},\ p_2'\in V\}>\varepsilon. \tag{A.2}$$

Let $U = B(p_2, \frac{\varepsilon}{4}) \cap V$. By Lemma 2.4, for the given $[p_1p_2]$ and any $p_2' \in U$

$$\exists!\ [p_1p_2'] \text{ such that } |\uparrow_{p_1}^{p_2}\uparrow_{p_1}^{p_2'}| = |p_2p_2'|. \tag{A.3}$$

Note that we need only to prove that $|\uparrow_{p_1}^{p_2^1}\uparrow_{p_1}^{p_2^2}| = |p_2^1p_2^2|$ for all $p_2^1, p_2^2 \in U$, where $\uparrow_{p_1}^{p_2^j}$ is the direction of the $[p_1p_2^j]$ found in (A.3). If this is not true, by Lemma 2.3 there is another minimal geodesic $[p_1p_2^2]'$ between p_1 and p_2^2 such that $|\uparrow_{p_1}^{p_2^1}(\uparrow_{p_1}^{p_2^2})'| = |p_2^1p_2^2|$. However, $|\uparrow_{p_1}^{p_2^2}(\uparrow_{p_1}^{p_2^2})'| \leq |p_2^2p_2| + |p_2p_2^1| + |p_2^1p_2^2| < \varepsilon$, which contradicts (A.2). $\square$

A.3. On the metric of $U_1 * U_2$ in (3.4.2)

Let X and Y be two Alexandrov spaces with curvature ≥ 1 (especially two Riemannian manifolds with sectional curvature ≥ 1). The canonical metric on the join space $X * Y \triangleq X \times Y \times [0, \frac{\pi}{2}]/\sim$, where $(x, y, t) \sim (x', y', t') \Leftrightarrow t = t' = 0$ and $x = x'$ or $t = t' = \frac{\pi}{2}$ and $y = y'$, is defined as follows ([BGP]):

$$\cos|p_1p_2| = \cos t_1 \cos t_2 \cos|x_1x_2| + \sin t_1 \sin t_2 \cos|y_1y_2|$$

(with $|p_1p_2| \leq \pi$) for any $p_i \triangleq [(x_i, y_i, t_i)] \in X * Y$. It can be proved ([BGP]) that $X * Y$ (endowed with such a metric) is also an Alexandrov space with curvature ≥ 1 and $\dim(X * Y) = \dim(X) + \dim(Y) + 1$ (especially, it is the unit sphere if both X and Y are unit spheres).

A.4. On the case where $n_1, n_2 > 0$ in the Main Theorem

Proposition A.4. *In the Main Theorem, if $n_1, n_2 > 0$, then either $M \overset{\text{iso}}{\cong} \mathbb{S}^n$ or $\mathbb{RP}^n$, or $M_1 = M_2^{\geq\frac{\pi}{2}}$ $(= M_2^{=\frac{\pi}{2}})$ and $M_2 = M_1^{\geq\frac{\pi}{2}}$ $(= M_1^{=\frac{\pi}{2}})$.*

Proof. Let $\bar{M}_1$ denote $M_2^{\geq\frac{\pi}{2}}$. It suffices to show that $M \overset{\text{iso}}{\cong} \mathbb{S}^n$ or $\mathbb{RP}^n$ if $M_1 \neq \bar{M}_1$. Note that $\bar{M}_1 = M_2^{=\frac{\pi}{2}}$ by Lemma 2.1. Then according to [RW] (cf. Footnote 5), $\dim(\bar{M}_1) = n_1 + 1$, and $\lambda_{q_1q_2} \equiv \bar{h} < +\infty$ for all $q_1 \in \bar{M}_1^\circ$ and $q_2 \in M_2$ (note that $\bar{M}_1$ may have nonempty boundary). Fix an arbitrary $[q_1q_2]$ with $q_1 \in \bar{M}_1^\circ$ and $q_2 \in M_2$, and consider $f_{[q_1q_2]} : M_2 \to (\Sigma_{q_1}\bar{M}_1)^{=\frac{\pi}{2}}$ $(= \mathbb{S}^{n_2})$. By (3.4.1), we conclude that $\sec_{M_2} \equiv 1$ or $n_2 = 1$, so $\lambda_{p_1p_2} \equiv h < +\infty$ for all $p_i \in M_i$ by Proposition 3.1. Hence, it follows from Lemma 3.3 and its proof that $M \overset{\text{iso}}{\cong} \mathbb{S}^n$ or $\mathbb{RP}^n$. $\square$

References

[B] A. L. Besse, Manifolds all of whose geodesics are closed, Ergebnisse Math. Grenzbebiete 93, Springer, 1978.

[BGP] Yu. Burago, M. Gromov and G. Perel′man, A.D. Alexandrov spaces with curvature bounded below, Uspeckhi Mat. Nank 47(2): 3-51, 1992.

[CE] J. Cheeger and D. G. Ebin, Comparison theorems in Riemannian geometry, New York, 1975.

[CG] J. Cheeger and D.Gromoll, On the lower bound for the injectivity radius of 1/4-pinched Riemannian manifolds, J. Diff. Geom., 15: 437-442, 1980.

[Fr] T. Frankel, Manifolds of positive curvature, Pacific J. Math., 11: 165-174, 1961.

[GG1] D. Gromoll and K. Grove, A generalization of Berger's rigidity theorem for positively curved manifolds, Ann. Scient. Ec. Norm. Sup., 20(4): 227-239, 1987.

[GG2] D. Gromoll and K. Grove, The low-dimensional metric foliations of Euclidean spheres, J. Diff. Geom., 28: 143-156, 1988.

[GM] K. Grove and S. Markvorsen, New extremal problems for the Riemannian recognition program via Alexandrov geometry, J. of AMS, 8(1): 1-28, 1995.

[GS] K. Grove and K. Shiohama, A generalized sphere theorem, Ann. of Math., 106: 201-211, 1977.

[P] P. Peterson, Riemannian Geometry, GTM 171, Springer-Verlag, 1998.

[RW] X. C. Rong and Y. S. Wang, Finite Quotient of Join Rigidity in Alexandrov Geometry, Preprint.

[Sa] R. H. Sady, Free involutions on complex projective spaces, Mich. Math. J., 24: 51-64, 1977.

[SSW] X. L. Su, H. W. Sun and Y. S. Wang, On $\frac{\pi}{2}$-separate subsets of Alexandrov spaces with curvature ≥ 1, arXiv: 1403.3169v2 [math. MG].

[SW] Z. Y. Sun and Y. S. Wang, On the radius of locally convex subsets in Alexandrov spaces with curvature ≥ 1 and radius $> \frac{\pi}{2}$, Front. Math. China, 9(2): 417-423, 2014.

[W] F. Wilhelm, The radius rigidity theorem for manifolds of positive curvature, J. Diff. Geom., 44: 634-665, 1996.

[Wi1] B. Wilking, Index parity of closed geodesics and rigidity of Hopf fibrations, Invent. Math., 144: 281-295, 2001.

[Wi2] B. Wilking, Torus actions on manifolds of positive sectional curvature, Acta Math., 191: 259-297, 2003.

[Ya] T. Yamaguchi, Collapsing 4-manifolds under a lower curvature bound, 2002.

School of Mathematical Sciences (and Lab. math. Com. Sys.), Beijing Normal University, Beijing, 100875 P.R.C.
e-mail: suxiaole@bnu.edu.cn; wyusheng@bnu.edu.cn

Mathematics Department, Capital Normal University, Beijing, 100037 P.R.C.
e-mail: 5598@cnu.edu.cn